Reflected diffusions and applications to finance and operations management

By

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Abstract

This dissertation provides explicit solutions to four special stochastic optimal control problems for reflected diffusions and Markov modulated reflected diffusions.

The main mathematical tool that we use is the ergodic theory for stochastic differential equations (SDE's), in particular, the ergodic theory for reflected diffusions. First, we present some basic definitions of reflected diffusions, the Itô's formulas. After this we study the ergodic theorems, namely, the law of large numbers, for reflected diffusions. The reflected diffusions and Markov modulated reflected diffusions are used to simulate bounded price dynamics regulated by policy makers. Motivated by this, we analyze several stochastic optimal control problems for reflected diffusions (or for Markov modulated reflected diffusions). These stochastic optimal control problems are solved in the sense that we can reduce them to some explicit optimization problems. Numerical computations are also given for the reduced optimization problems. Our work resulted in one accepted paper, one completed paper and two on-going projects.

This dissertation is divided into eight chapters. Chapter 1 provides a concise summary of this dissertation and motivations of our research. Chapters 2 and 3 provide some necessary background material on reflected diffusions. Chapter 4 presents similar results for Markov modulated reflected diffusions based on the paper [28].

The content of Chapter 5 is our paper "Optimal pricing barriers in a regulated market using reflected diffusion processes". This paper has been published by Quantitative

Finance (online version is available). In this paper, a class of one-dimensional reflected diffusions is used to simulate bounded price dynamics regulated by a policy maker, and the goal is to determine the optimal pricing ceiling and floor for the ergodic governing cost. We consider the running cost associated with the deviation of the process from the desired target level, and also the control cost from the interventions in an effort to keep the process inside the boundaries. Both ergodic cost and an infinite horizon discount cost are studied. Numerical examples are provided to illustrate our main results.

Chapter 6 presents a ready-to-submit project on a stochastic optimal control problem for Markov modulated reflected diffusions. This is an extension of our previous published work to a reflected diffusions with Markov switching. The extension of the model is important since it covers more sophisticated situations. But the explicit solutions are more difficulty to obtain. We use some recent results of [28] to obtain explicit solutions.

Chapter 7 contains an on-going project on asymptotically optimal control of a queueing model. Inspired by [25], we consider a queueing model (again a reflected diffusion process) which allows the impatient customers to renege during the waiting. An iterative strategy is developed to locate the optimal admission barrier, and properties of optimal barrier are derived, with two concrete examples.

In general stochastic control problems are hard to solve explicitly and the simple solvable models are linear quadratic control (linear system and quadratic performance functional). The last chapter (Chapter 8) exhibits another on-going project of finding the optimal control policy for one linear-quadratic problems for reflected diffusions. The analytic results and procedure towards the optimal control policy are shown, and we are currently working on the applications associated with this model.

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Contents

Chapter 1

Introduction

This dissertation concerns with search of explicit solutions of some stochastic optimal control problems for reflected stochastic differential equations (SDE's). The problems are motivated by their applications to finance and management science. The topics contain the optimal barrier problems for reflected diffusions, or Markov-modulated reflected diffusions. A linear quadratic control problem is also solved.

The dissertation consists of one published paper, one ready-to-submit paper, and two on-going projects, jointly with my advisors Yaozhong Hu and Chihoon Lee. They are listed as follows:

(1) (included in Chapter 5)

Han, Z.; Hu, Y.; Lee, C. Optimal pricing barriers in a regulated market using reflected diffusion processes. Quantitative Finance, accepted.

(2) (included in Chapter 6)

Han, Z.; Hu, Y.; Lee, C. Optimal barrier for some Markov modulated diffusions.

(3) (included in Chapter 7)

Han, Z.; Hu, Y.; Lee, C. Asymptotically optimal control for a queueing model with impatient costumers.

(4) (included in Chapter 8)

Han, Z.; Hu, Y.; Lee, C. Optimal linear-quadratic regulator for reflected diffusions.

Before the presentation of our main results the first three chapters (Chapter 2, 3 and 4) of this dissertation will provide some brief background materials for reflected diffusions and Markov-modulated reflected diffusions along with ergodic theory for SDE's. A list of references is provided at the end of the dissertation.

Next we explain the motivation of our applications in more details, with an emphasis on Chapter 5 and 6.

It is well known that some basic capital and consumption goods, as well as key economic variables are controlled by or can be influenced by the regulating authority (the government body or the central bank). For instance, the usual house living and elementary industry utilities like gas, water and electricity, as well as some fundamental economic variables such as domestic interest rates, foreign exchange rates, and major stock indexes. In these cases, the regulating authority often has the power to set a pair of pricing ceiling and floor, in order to keep the entire national economic environment healthy and protected. Thus, this situation naturally triggers a question to ask:

What is the best pricing ceiling and pricing floor?

In an effort to answer this question, we first notice that the reflected Stochastic Differential Equation (reflected diffusion) model arises as an important approximating price model, since it is ideal for the purpose of describing the bounded price dynamics mentioned above.

Secondly, our objective function should be a cost structure integrating both running cost and interventions cost. Let's illustrate the rationale behind this by one specific example: the foreign exchange rate. For many countries, it is their central bank's responsibility to guarantee its currency exchange rate to stay in a normal range (often called the target zone), and thus keeps a stable national economic climate. Once the exchange rate hits the boundaries of target zone, the central bank will exercise its financial power to intervene in the foreign exchange market as to force the exchange rate back. This way comes the interventions cost. However, it is generally accepted that, in countries that try to keep their exchange rate within a target zone, most interventions in the foreign exchange rate market take place inside the zone (see Avesani and Gallo [1] and Bertola and Caballero [2] for empirical evidence). The explanation of this phenomenon is that the pressure on the exchange rate increases when the exchange rate is at the boundary, since speculative attacks are more likely to be triggered and as a result the cost of saving the exchange rate increases. A large literature has discussed this topic (see Cadenillas and Zapatero [9] to access more details) once the boundary is reached. And thus a running cost happened inside the target zone should be added to the objective cost structure. Therefore, how to decide the optimal boundaries which minimizes the total cost structure would be an important and necessary research question to ask.

Another example of such bounded dynamics would be the daily price fluctuation limits in stock markets, namely, the percentage or amount that securities values are allowed to rise or fall during a trading day. For instance, the Chinese stock market has a 10 percentage daily up limit and down limit for every listed sock in the exchange markets. Once the limit is touched by a security, the exchange's management institutes will suspended all trading activities on this security, for the sake of discouraging excessive volatility. And this is especially important for futures contracts and options, which are almost always volatile. If the daily price limit is set too small then it will restrain the healthy development of listed companies, and on the other hand a too large limit cannot function as a sound restriction method for harmful volatility. Thus again the answer to the question how to determine the optimal daily price limits is important.

In addition, a large amount of literature provides strong evidence for the existence of regime changes. In particular, in history unusual high volatility in U.S. short-term interest rates has occurred during the episodes of the 1973 and 1979 OPEC oil crises, the 1979-82 Federal Reserve Monetary Experiment, and the 1987 stock market crash. We refer our reader to [10] for further supporting materials. Thus, we employ Markov-modulated (regime-switching) diffusions in our second project, leading to a more generalized model. For a general theory of Markov switching processes we refer to [29] and the references therein.

Chapter 2

One dimensional reflected diffusions and properties

2.1 Definitions

The idea about reflected diffusion originated from techniques for solving PDEs.

Roughly speaking, a one-dimensional reflected diffusion process behaves like a regular diffusion in the interior of its domain (a,b) until it reaches the boundary $\{a,b\}$, and as soon as the sample path hits the boundary, it returns to the interior in a manner that the "regulating" force to the interior is minimal.

Given a filtered probability space $\Lambda := (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with the filtration $(\mathscr{F}_t)_{t\geq 0}$, the reflected diffusion $\{X_t : t \geq 0\}$ with two-sided barriers a and b is defined as:

$$\begin{cases} dX_t = -f(X_t)dt + \sigma(X_t)dB_t + dL_t - dR_t, \\ X_0 = x \in [a,b]. \end{cases} \tag{1.1}$$

Here, σ is strictly positive and Lipschitz continuous, and f is Lipschitz continuous. a, b with $-\infty \leq a < b \leq \infty$ are given real numbers, and $(B_t, 0 \leq t < \infty)$ is the one-dimensional standard Brownian motion on Λ.

The processes $L = (L_t)_{t\geq 0}$ and $R = (R_t)_{t\geq 0}$ are the minimal non-decreasing and non-negative processes, which restrict the process X_t to be inside $[a,b]$ for all $t \geq 0$. More precisely, the processes $\{L_t, t \geq 0\}$ and $\{R_t, t \geq 0\}$ increase only when X_t hits the boundary a and b, respectively, so that $L_0 = R_0 = 0$ and

$$\int_0^\infty I_{\{X_t > a\}} dL_t = 0 \quad \text{and} \quad \int_0^\infty I_{\{X_t < b\}} dR_t = 0\,.$$

By the nature of the so-called "reflected" diffusion processes, people have extensively studied its applications in the field of operational management, such as inventory management, queueing models, production lines/cash flow management, service engineering systems, as well as in the field of regulated financial market, like governmental macro-interventions on processes of basic goods, on the domestic interest rates, etc.

2.2 Properties and Itô's formula

Let the state space of $\{X_t : t \geq 0\}$ be $S = (a,b)$ with $-\infty \leq a < b \leq \infty$. On the set $B(S)$ of all real-valued, bounded, Borel measurable function ϕ define the transition operator

$$T_t\phi(x) := E(\phi(X_t)|X_0 = x) = \int \phi(y)p(t;x,dy)\,,$$

where $p(t;x,dy)$ is the transition probability density of the process. And the infinitesimal generator A of the Markov process X_t is defined by

$$A\phi(x) := \lim_{t\downarrow 0} \frac{T_t\phi(x) - \phi(x)}{t}.$$

Next it can be proved easily by Taylor's expansion that

$$A\phi(x) = \frac{1}{2}\sigma^2(x)\phi''(x) - f(x)\phi'(x) \tag{2.1}$$

for our reflected diffusion process $\{X_t : t \geq 0\}$.

Now let's study Itô's formula for reflected diffusion process. Given a twice continuously differentiable function $g(x)$:

$$\begin{aligned} g(X_t) &= g(x) + \int_0^t g'(X_s)dX_s + \frac{1}{2}\int_0^t g''(X_s)d\langle X\rangle_s \\ &= g(x) + \int_0^t g'(X_s)(-f(X_s)ds + \sigma(X_s)dB_s + dL_s - dR_s) \\ &\quad + \frac{1}{2}\int_0^t g''(X_s)\sigma^2(x)ds \end{aligned} \tag{2.2}$$

Then apply (2.1) and recall that by definitions L_t and R_t only increase when X_t hits a and b respectively, we can get

$$g(X_t) = g(x) + \int_0^t Ag(X_s)ds + \int_0^t g'(X_s)\sigma(X_s)dB_s + g'(a)L_t - g'(b)R_t \tag{2.3}$$

And for more details about properties of reflected processes, one can check [13].

Chapter 3

Ergodic theory for diffusions

3.1 Definitions

A Markov process X_t on the state space $S = (a,b)$ with $-\infty \leq a < b \leq \infty$ is said to be a diffusion with drift coefficient $\mu(x)$ and diffusion coefficient $\sigma^2(x) > 0$, if it has continuous sample paths, and:

$$\begin{cases} E(X_{s+t} - X_s | X_s = x) = \mu(x)t + o(t), \\ E((X_{s+t} - X_s)^2 | X_s = x) = \sigma^2(x)t + o(t), \\ E(|X_{s+t} - X_s|^3 | X_s = x) = o(t), \end{cases} \tag{1.1}$$

as $t \downarrow 0$ for every $x \in S$.

Let $\mu(x), \sigma(x)$ be continuously differentiable, with bounded derivatives on $(-\infty, \infty)$. Assume σ'' exists and is continuous. Let $p(t;x,dy)$ be the transition probability distribution of corresponding Markov process. Again as in the previous chapter, on the set $B(S)$ of all real-valued, bounded, Borel measurable function f on S we define the transition operator

$$T_t f(x) := E(f(X_t)|X_0 = x) = \int f(y)p(t;x,dy)\,.$$

The infinitesimal generator A of the Markov process X_t is defined by

$$Af(x) := \lim_{t\downarrow 0} \frac{T_t f(x) - f(x)}{t}\,.$$

Let $I(x,z) = \int_x^z \frac{2\mu(y)}{\sigma^2(y)}dy$. Fix $x_0 \in S$. Define

$$s(x) = \int_{x_0}^{x} e^{-I(x_0,z)}dz, \quad \text{and} \quad m(x) = \int_{x_0}^{x} \frac{2}{\sigma^2(z)} e^{I(x_0,z)}dz.$$

For simplicity we write the conditional probability $P(E|X_0 = x)$ as $P_x(E)$ for any event E. Let $\rho_{xy} = P_x(\ \exists\, t_0 \geq 0$, such that $X_{t_0} = y\ )$, $x,y \in S$, i.e., the probability of X_t starting from x ever reaches y.

A state y is recurrent if $\rho_{xy} = 1$ for all $x \in S$ such that $\rho_{yx} > 0$. If all states in S are recurrent, then the diffusion is said to be recurrent.

A diffusion on S is positive recurrent if $E_x\tau_y < \infty$ for all $x,y \in S$, where $\tau_y = inf\{t \geq 0, X_t = y\}$. A recurrent diffusion that is not positive recurrent is said to be null recurrent.

3.2 Strong Law of Large Numbers for diffusions

It is straightforward to check that by definition

$$Af(x) = \frac{d}{dm(x)}\left(\frac{df(x)}{ds(x)}\right).$$

The following conclusions can be derived.

Proposition 3.1. *Suppose $S = (a,b)$.*

A diffusion with coefficients $\mu(x)$, $\sigma^2(x)$ is recurrent, if and only if $s(a) = -\infty$ and $s(b) = \infty$.

A diffusion is positive recurrent, if and only if $s(a) = -\infty$, $m(a) > -\infty$, and $s(b) = \infty$, $m(b) < \infty$.

Here we provide the outline of proof, and for more details one can check [3]: Let $[c,d] \subset S$. Define $(X_h^+)_t = X_{h+t}$, $\tau' = \tau - h$ with $\tau = \tau_c \wedge \tau_d$. Clearly the event $\{\tau > h\}$ is determined by $\{X_u : 0 \le u \le h\}$. Let $\phi(x) := P(\{X_t^x\}$ hits c before d) .

Notice that

$$\begin{aligned} P_x(\tau > h, X_\tau = c) &= \mathbb{E}_x\left(P(\tau > h, X_\tau = c | \{X_u : 0 \le u \le h\})\right) \\ &= \mathbb{E}_x\left(P(\tau > h, (X_h^+)_{\tau'} = c | \{X_u : 0 \le u \le h\})\right) \\ &= \mathbb{E}_x\left(1_{\{\tau > h\}}\phi(X_h)\right), \end{aligned} \tag{2.1}$$

where the last step follows from the strong Markov property of X_t. ϕ can be extended smoothly over $x < c$ and $x > b$, and vanishes outside a compact set in S. Thus we can go further from (2.1):

$$\begin{aligned} P_x(\tau > h, X_\tau = c) &= \mathbb{E}_x\phi(X_h) - \mathbb{E}_x\left(1_{\{\tau \le h\}}\phi(X_h)\right) \\ &= \int_S \phi(y)p(h;x,y)dy + o(h), \end{aligned} \tag{2.2}$$

as $h \downarrow 0$ by the definition of diffusions from (1.1). Also note that

$$P_x(\tau \le h, X_\tau = c) \le P_x(\tau \le h) = o(h), \quad \text{as } h \downarrow 0. \tag{2.3}$$

Therefore, by the above (2.2) and (2.3), we can deduce that

$$\phi(x) = P_x(X_\tau = c) = T_h\phi(x) + o(h), \quad \text{as } h \downarrow 0.$$

Now by definition of infinitesimal generator we get

$$A\phi(x) = \lim_{h \downarrow 0} \frac{T_h\phi(x) - \phi(x)}{h} = 0$$

Consequently the probability $\phi(x) = P(\{X_t^x\}$ hits c before d) is the solution to the following ordinary differential equation:

$$\begin{cases} A\phi(x) = \dfrac{1}{2}\sigma^2(x)\phi''(x) + \mu(x)\phi'(x) = 0 \\ \phi(c) = 1, \quad \phi(d) = 0 \end{cases} \tag{2.4}$$

Since $Af(x) = \frac{d}{dm(x)}\left(\frac{df(x)}{ds(x)}\right)$, by integrating twice and using the boundary conditions we can get

$$\phi(x) = \frac{s(d) - s(x)}{s(d) - s(c)}.$$

Recall that $\rho_{xy} = P_x(X_t$ ever reaches $y)$, it's straightforward to see that if $s(a) = -\infty$, then $\rho_{xy} = 1$ for any $y > x$ (by letting $d = y$, and $c \downarrow a$); and if $s(b) = \infty$, then $\rho_{xy} = 1$ for any $y < x$ (by letting $c = y$, and $d \uparrow b$).

Now define $M(x) = E_x\tau$. Using similar idea it can be shown that $M(x)$ is the solution to

$$\begin{cases} AM(x) = \frac{1}{2}\sigma^2(x)M''(x) + \mu(x)M'(x) = -1 \\ M(c) = M(d) = 0 \end{cases} \tag{2.5}$$

Once again by $Af(x) = \frac{d}{dm(x)}\left(\frac{df(x)}{ds(x)}\right)$, we can get

$$M(x) = \frac{s(x)}{s(d)}\int_c^d m(y)ds(y) - \int_c^x m(y)ds(y).$$

If b is inaccessible ($\tau_b = \infty$), then $E_x\tau_c = \lim_{d\uparrow b} E_x\tau$.

It can be shown that if S has an inaccessible upper end point b and $s(b) = \infty$, then $E_x\tau_c < \infty$ if and only if $m(b) < \infty$. Proceeding in the same manner, it follows that if S has an inaccessible lower end point a and $s(a) = -\infty$, then $E_x\tau_d < \infty$ if and only if $m(a) > -\infty$. Now we are ready for the main theorem.

Theorem 3.2. *Suppose the diffusion is positive recurrent on $S = (a,b)$.*

(1) Then there exists a unique invariant distribution $\pi(dx)$.

(2) For every real-valued f such that $\int_S |f(x)|\pi(dx) < \infty$,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X_s)ds = \int_S f(x)\pi(dx)$$

with probability 1.

(3) If the end points a, b of S are inaccessible, then the invariant probability has a density $\pi(x)$, which is the unique normalized integrable solution to $\frac{1}{2}\frac{d}{dx}(\sigma^2(x)\pi(x)) - \mu(x)\pi(x) = 0$. And $\pi(x) = \frac{m'(x)}{m(b)-m(a)}$.

Again let us provide the outline of proof here:

Pick $x_0, y_0 \in S$ and let $\eta_0 = 0$, $\eta_1 = inf\{t \geq 0, X_t = x_0\}$, $\eta_{2r} = inf\{t \geq \eta_{2r-1}, X_t = y_0\}$,

$\eta_{2r+1} = inf\{t \geq \eta_{2r}, X_t = x_0\}$, for $r = 1, 2, \cdots$ Define

$$Z_r = \int_{\eta_{2r}}^{\eta_{2r+2}} f(X_s)ds.$$

By strong Markov property one can show that $\{Z_r\}$ is i.i.d. Using positive recurrence and strong law of large numbers we can get as $r \to \infty$,

$$\begin{cases} \dfrac{\eta_{2r}}{r} = \dfrac{\sum_{r'=1}^{r}(\eta_{2r'} - \eta_{2(r'-1)})}{r} \to E_{y_0}\eta_2, \\ \dfrac{1}{r}\displaystyle\sum_{r'=1}^{r} Z_{r'-1} \to E_{y_0}Z_0 = E_{y_0}\int_0^{\eta_2} f(X_s)ds. \end{cases} \tag{2.6}$$

By $\lim\limits_{r\to\infty} \frac{1}{\eta_{2r}} \int_0^{\eta_{2r}} f(X_s)ds = \lim\limits_{t\to\infty} \frac{1}{t} \int_0^t f(X_s)ds$ (for every f s.t. $E_{y_0} \int_0^{\eta_2} |f(X_s)|ds < \infty$), one gets

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X_s)ds = \frac{1}{E_{y_0}\eta_2} E_{y_0} \int_0^{\eta_2} f(X_s)ds. \tag{2.7}$$

Now in case $f = 1_B$, where B is a Borel subset of S, we define

$$\pi(B) := \frac{1}{E_{y_0}\eta_2} E_{y_0} \int_0^{\eta_2} 1_B(X_s)ds. \tag{2.8}$$

Starting from simple functions, one can see that

$$\frac{1}{E_{y_0}\eta_2} E_{y_0} \int_0^{\eta_2} f(X_s)ds = \int_S f(x)\pi(dx)$$

holds for all f satisfying $E_{y_0} \int_0^{\eta_2} |f(X_s)|ds < \infty$. Combine the above results one has

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X_s)ds = \int_S f(x)\pi(dx). \tag{2.9}$$

To prove π is the invariant probability, take expectations on both sides of (2.9), one gets

$$\int_S f(x)\pi(dx) = \lim_{t\to\infty}\frac{1}{t}\int_0^t E_y f(X_s)ds,$$

and

$$\begin{aligned}\int_S f(x)\pi(dx) &= \lim_{t\to\infty}\frac{1}{t}\int_0^t E_y f(X_s)ds = \lim_{t\to\infty}\frac{1}{t}\int_0^t T_s f(y)ds \\ &= \lim_{t\to\infty}\frac{1}{t}\int_h^{t+h} T_s f(y)ds = \lim_{t\to\infty}\frac{1}{t}\int_0^t T_{s+h} f(y)ds \\ &= \lim_{t\to\infty}\frac{1}{t}\int_0^t T_s T_h f(y)ds = \int_S T_h f(x)\pi(dx).\end{aligned} \tag{2.10}$$

Thus one obtains $\int_S T_h f(x)\pi(dx) = \int_S f(x)\pi(dx)$. Take $f = 1_B$, then $T_h f(x) = P(h;x,B)$. Following this one has

$$\begin{aligned}P_\pi(X_h \in B) &= E_\pi P(X_h \in B|X_0) = \int_S P(h;x,B)\pi(dx) \\ &= \int_S 1_B(x)\pi(dx) = \pi(B) = P_\pi(X_0 \in B).\end{aligned} \tag{2.11}$$

Uniqueness can be proved by the definition of invariant distribution. To prove Part (c), one can directly check that $\pi(x) = \frac{m'(x)}{m(b)-m(a)}$ is the invariant distribution and satisfies the corresponding ODE. Hence the proof.

Chapter 4

Markov-modulated reflected diffusions and ergodic property for one special type

4.1 Definitions

In general, the state of a Markov modulated reflected diffusion process consists of two components. One of them describes the continuous dynamics, and the other displays the discrete state. The continuous dynamics are diffusion processes, while the discrete state is driven by a Markov chain representing all possible regimes. Quite a lot literature has shown the existence of regime changes in real-world economic activities. For example, the fluctuations of the U.S. short-term interest rates were significantly sharper during the episodes of the 1973 and 1979 OPEC oil crises, the 1979-82 Federal Reserve Monetary Experiment, and the 1987 stock market crash. For more supporting materials and examples, we refer the reader to [10] and the references therein.

Let $\{J_t, t \geq 0\}$ be an irreducible and recurrent continuous-time Markov chain with finite state space $\mathscr{S}$. Suppose the intensity matrix of J_t is $Q = (q_{ij})_{i,j\in\mathscr{S}}$. Now the Markov-modulated reflected diffusion process $\{X_t, t \geq 0\}$ is defined to be the solution of the following stochastic differential equation:

$$\begin{cases} dX_t = b(J_t, X_t)dt + \sigma(J_t, X_t)dB_t + dL_t - dR_t, \quad t \geq 0, \\ X_0 = x \in [k, K], \end{cases} \tag{1.1}$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion, k and K are real numbers, b and σ are some given functions from $\mathscr{S} \times \mathbb{R} \to \mathbb{R}$. And L_t and R_t are the minimal non-decreasing processes that restrict X_t within the interval $[k, K]$, as in the definitions from Chapter 2.

4.2 Ergodic property for one special case

Now using the results in [28], we show the derivation of the Laplace transform of the stationary distribution of one special type of X_t as follows:

$$\begin{cases} dX_t = -(b(J_t) + \lambda(J_t)X_t)dt + \sigma(J_t)dB_t + dL_t, \quad t \geq 0, \\ X_0 = x \in [0, \infty], \end{cases} \tag{2.1}$$

where σ, λ and b are strictly positive functions defined on $\mathscr{S}$. L_t is the regulating process keeps $X_t \geq 0$ and other notations are as defined before. Now let $g_i(x)$ be the stationary density of (X_∞, J_∞) at (x, i). Define $\hat{g}_i(\alpha) = \int_0^\infty e^{-\alpha x} g_i(x)dx$, the Laplace transform of g_i, and $\hat{g}(\alpha) = (\hat{g}_1(\alpha), \hat{g}_2(\alpha), ..., \hat{g}_n(\alpha))^T$. And for simplicity we denote $\sigma_i = \sigma(i)$, $\lambda_i = \lambda(i)$ and $b_i = b(i)$. Then we shall present the main theorem form [28] below:

Theorem 4.1. *Assume that the stationary distribution of (X_t, J_t) exists. Then $\hat{g}(\alpha)$ satisfies the following differential equation:*

$$A(\alpha)\hat{g}(\alpha) - B(\alpha)\hat{g}'(\alpha) = \alpha P, \tag{2.2}$$

where $A(\alpha) = Q^T + \mathrm{diag}(\sigma_i^2\alpha^2/2 + b_i\alpha)$*,* $B(\alpha) = \mathrm{diag}(\lambda_i\alpha)$*, and* P *is a constant independent of* α*.*

Proof We begin with the infinitesimal generator L of (X_t, J_t). Assume f is bounded and twice continuously differentiable in x. By Itô's formula,

$$
\begin{aligned}
Lf(x,i) &= \lim_{t\to 0} \frac{\mathbb{E}_{(x,i)} f(X_t,J_t) - \mathbb{E}_{(x,i)} f(X_0,J_0)}{t} \\
&= \lim_{t\to 0} \frac{\mathbb{E}_{(x,i)} f(X_t,J_t) - \mathbb{E}_{(x,i)} f(X_t,J_0) + \mathbb{E}_{(x,i)} f(X_t,J_0) - \mathbb{E}_{(x,i)} f(X_0,J_0)}{t} \\
&= \lim_{t\to 0} t^{-1}((1+q_{ii}t)\mathbb{E}_{(x,i)} f(X_t,i) + \sum_{i\neq j} q_{ij}t\mathbb{E}_{(x,i)} f(X_t,j) - \mathbb{E}_{(x,i)} f(X_t,i) \\
&- \mathbb{E}_{(x,i)} \int_0^t (b(J_s) + \lambda(J_s)X_s) f'(X_s,J_0)ds + f'(0,J_0)\mathbb{E}_{(x,i)} L_t \\
&+ \mathbb{E}_{(x,i)} \int_0^t \sigma(J_s) f'(X_s,J_0)dB_s + \frac{1}{2}\mathbb{E}_{(x,i)} \int_0^t \sigma^2(J_s) f''(X_s,J_0)ds) \\
&= \sum_j q_{ij} f(x,j) - b_i f'(x,j) - \lambda_i x f'(x,i) + \frac{1}{2}\sigma_i^2 f''(x,i) + f'(0,i)\lim_{t\to 0}\mathbb{E}_{(x,i)} \frac{L_t}{t}.
\end{aligned}
\tag{2.3}
$$

Notice that $g_i(x)$ solves

$$\sum_{i=1}^n \int_0^\infty (Lf)(x,i) g_i(x)dx = 0.$$

Setting $f(x,i) = (e^{-\alpha x} - 1)h(i)$ where h is a bounded function on $\mathscr{S}$, by the above two results we can get

$$\sum_{i=1}^n h(i) \int_0^\infty H_i(x)dx = 0,$$

where

$$
\begin{aligned}
H_i(x) = &\sum_{j=1}^n q_{ij}(e^{-\alpha x} - 1)g_j(x) + b_i\alpha e^{-\alpha x} g_i(x) + \lambda_i \alpha x e^{-\alpha x} g_i(x) \\
&+ \frac{1}{2}\sigma_i^2\alpha^2 e^{-\alpha x} g_i(x) - \alpha g_i(x)\lim_{t\to 0}\mathbb{E}_{(x,i)}\frac{L_t}{t}.
\end{aligned}
\tag{2.4}
$$

Now define $l_x^i = \lim_{t\to 0} \mathbb{E}_{(x,i)} \frac{L_t}{t}$ and $p_i = \int_0^\infty l_x^i g_i(x)dx$. Note that by definitions of $\hat{g}_i$ the above yields

$$\sum_{i=1}^{n} h(i) \left[\sum_{j=1}^{n} q_{ij}\hat{g}_j(\alpha) + b_i\alpha\hat{g}_i(\alpha) - \lambda_i\alpha\hat{g}_i'(\alpha) + \frac{1}{2}\sigma_i^2\alpha^2\hat{g}_i(\alpha) - \alpha p_i \right] = 0.$$

Thus let $P = (p_1, ..., p_n)^T$, and since h is arbitrary, we can get

$$A(\alpha)\hat{g}(\alpha) - B(\alpha)\hat{g}'(\alpha) = \alpha P.$$

Hence equipped with the Laplace transforms from the above theorem, one is able to compute all the moments of the stationary distribution for 2.1.

Chapter 5

Optimal pricing barriers in a regulated market using reflected diffusion processes

In this chapter, we consider a class of one-dimensional reflected stochastic differential equations (SDEs). Such reflected SDE models arise as the key approximating processes in a regulated financial market system, and our main goal is to determine the set of optimal pricing barriers. We consider the running cost associated with the deviation of the process from the desired target level, and also the control cost from the interventions in an effort to keep the process inside the boundaries. Both a long time average (ergodic) cost criterion and an infinite horizon discount cost criterion, where the discount factor is allowed to vary from one period to another, are studied with numerical examples illustrating our main results.

5.1 Introduction

We consider the problem of finding *optimal* barriers for a class of one-dimensional reflected stochastic differential equations (SDEs). Roughly speaking, the solution of reflected SDE behaves like a solution of a SDE (with no reflection) in the interior of its domain (a,b) until it reaches the boundary $\{a,b\}$, and as soon as the sample path hits the boundary, it returns to the interior in a manner that the "regulating" force to the interior is minimal. Such reflected SDE models have been extensively studied in the field of operational management with stochastic flow systems, e.g., queueing networks, production lines/cash flow management, service engineering systems, etc. (cf. [13, 27, 19]). On the other hand, our main interest in this model stems from the fact that the reflected SDE model arises as an important approximating process in a *regulated* (or *controlled*) financial market system. For instance, the government would like to implement its macro-interventions on the prices of some basic goods and services, as well as the foreign exchange rates and the domestic interest rates. Therefore, the resulting price dynamics are controlled by the price ceiling and the price floor. For more motivations and nice practical examples related to the reflected SDEs and their applications in financial context, we refer the reader to a series of recent papers [4, 5, 6, 7] and also the references therein such as [18]. Our main goal in this paper lies in quantifying the set of optimal pricing barriers by analyzing reflected SDE models.

Let us now introduce the reflected SDE model more precisely. Given a filtered probability space $\Lambda := (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with the filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfying the *usual* conditions, we are concerned with the strong solution $\{X_t : t \geq 0\}$ of the following reflected SDE with two-sided barriers a and b (whose existence is guaranteed by an extension of the results of [21]):

$$\begin{cases} dX_t = -f(X_t)dt + \sigma dB_t + dL_t - dR_t, \\ X_0 = x \in [a,b]. \end{cases} \tag{1.1}$$

Here, $\sigma > 0$, $0 \leq a < b < \infty$ are given real numbers and $f : \mathbb{R} \to \mathbb{R}_+$ is a given Lipschitz continuous function and not identically zero on $[a,b]$, and $(B_t, 0 \leq t < \infty)$ is the one-dimensional standard Brownian motion on Λ. (We shall consider the more general state-dependent diffusion coefficient model in Section 5.3.) The processes $L = (L_t)_{t\geq 0}$ and $R = (R_t)_{t\geq 0}$ are the minimal non-decreasing and non-negative processes, which make the process $X_t \in [a,b]$ for all $t \geq 0$. More precisely, the processes L and R increase only when X hits the boundary a and b, respectively, so that $L_0 = R_0 = 0$ and

$$\int_0^\infty I_{\{X_t>a\}} dL_t = 0 \quad \text{and} \quad \int_0^\infty I_{\{X_t<b\}} dR_t = 0\,,$$

where $I(\cdot)$ is the indicator function. If $f(x) \equiv \mu \in (0,\infty)$, then (1.1) is reduced to the reflected Brownian motion with a nonnegative drift, and if $f(x) \equiv x$, the model (1.1) becomes the reflected Ornstein-Uhlenbeck process; both models are widely applicable in modeling and controlling for a class of queueing systems and storage models (cf. [26] and the references therein). In [6], the authors considered the model (1.1) when $f(x) \equiv x$ and $f(x) \equiv x^2$ (and $[a,b] \equiv [0,1]$) as two applications in a regulated financial market, that is, the price dynamics of the reference goods or service, and the risk-neutral term structure of the interest rate model (related to the price for a certain digital option), respectively.

Given the above reflected diffusions, we introduce our cost structure. For a set of controllable lower and upper barriers $(a,b), a < b$, the controller (e.g., government regulation body) is faced with a cost structure consisting of the following two additive components: during a time interval $[t,t+dt]$,

(i) a state-dependent running cost $h(X_t)dt$, and

(ii) a cost of $\alpha dL_t + \beta dR_t$ for interventions at the boundaries.

Here, $\alpha > 0, \beta > 0$ are constants, h is a non-negative function satisfying some basic assumptions. The running cost of (i) represents the cost due to the deviation of the state process from the desired target level set by the controller (such a level can be assumed to be zero without loss of generality since one could consider the *centered* process otherwise). The cost in (ii) represents the underlying intervention costs associated with the price floor and the price ceiling, with a fixed (possibly different) proportionality factor (i.e., α and β) at each boundary. We envision the scenario that the controller intervenes in the system in a *minimal* way. For instance, under a stable economy, the government would like to intervene in the market minimally; it uses only as much control/regulation as necessary so that the price process gets immediately to the nearest value on the upper/lower boundary. Such an action can be implemented by buying or selling related goods, or externally adjusting capital flows, whence the terms α and β would represent the associated transaction cost rates or, related interest rates.

We shall consider two types of control problems with the above cost structure. First problem (Problem 1) is concerned with minimizing a long time average (known as ergodic) cost criterion, while the second problem (Problem 2) extends the former to the finite or infinite time horizon discount cost where the discount factor is allowed to vary from one period (e.g., day, week, month) to another. To the best of our knowledge, such time-varying discount factor has not been widely considered in the literature. Also, notice that in our control problems the barriers a and b are the control variables and all other data are assumed to be known and fixed; this is a singular control problem (cf. [15]) as opposed to an impulse control problem considered in the literature. Problem 1 is studied in Section 5.2 via ergodic properties of the state process, and Problem 2 in Section 5.3 is analyzed based on the spectral expansion of the transition density function and identifying the solutions of the related Sturm-Liouville differential equations. We provide associated numerical examples and concrete computational schemes for each problem to illustrate our main results. Section 5.4 concludes with some discussion and remarks on the further work.

5.2 Ergodic barrier controls

Problem 1 Our first problem is to find the optimal lower and upper barriers a and b to minimize the following ergodic cost

$$J_1(a,b) = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T h(X_t)dt + \alpha L_T + \beta R_T\right], \tag{2.1}$$

where $\alpha, \beta > 0$ are given constants and h is given bounded and integrable function on $[a,b]$ and X_t, L_t and R_t are defined by (1.1). Problems of these characteristics are related to examples of singular control literature (cf. [15] and Chapter VIII of [11]). Although it is relatively simple, an analysis of Problem 1 serves as a roadmap for the results that need to be established in order to solve the more complex second Problem 2 for the finite/infinite horizon cost with time varying discount factors. We shall solve Problem 1 by finding the "explicit" form of $J_1(a,b)$. The proof of the following proposition is based on the ergodicity of $\{X_t\}$, Itô's formula for reflected process, and solving a class of second order differential equations with boundary conditions; its detailed presentation is deferred to the Appendix.

Proposition 5.1. *The ergodic cost can be expressed by*

$$J_1(a,b) = \left(\int_a^b h(x)e^{-\frac{2}{\sigma^2}\int_a^x f(r)dr}dx + \frac{\sigma^2}{2}\left(\alpha + \beta e^{-\frac{2}{\sigma^2}\int_a^b f(r)dr}\right)\right)\left(\int_a^b e^{-\frac{2}{\sigma^2}\int_a^u f(r)dr}du\right)^{-1}. \tag{2.2}$$

Thus our Problem 1 is now reduced to finding the optimal values a and b which minimize $J_1(a,b)$ given as in (2.2). We remark that the more involved case in which α and β may depend on a and b can be treated as well. For example, the higher the upper reflecting barrier b is, the penalty cost β associated with this price ceiling could be more severe. Mathematically, it is evident that our arguments (described in the Appendix, Section A) work as well when α and β are functions of a and b.

For the sake of concreteness and motivated from the models considered in [6] (e.g., the price dynamics, the risk-neutral term structure of the interest rate associated with a digital option application), we study the following cases. Take the drift function $f(x) = \theta x^n$ and the running cost function $h(x) = (x-\ell_0)^m$, where n, m and θ, ℓ_0 are some positive real numbers. Without loss of generality, we take $\ell_0 = 1$ henceforth. We shall start with the simplified situation when the lower barrier $a = 0$. In this setting, the ergodic cost function becomes $J_1(b) \equiv J_1(0,b)$, which is clearly twice continuously differentiable with respect to b. Indeed, we can verify that

$$J_1'(b) = (\gamma(b))^{-1} e^{-\frac{2\theta}{(n+1)\sigma^2} b^{n+1}} H(b),$$

where $\gamma(b) = \int_0^b e^{-\frac{2\theta}{(n+1)\sigma^2} x^{n+1}} dx$ and

$$H(b) = (b-1)^m - \theta\beta b^n - \frac{1}{\gamma(b)} \left(\int_0^b (x-\ell_0)^m e^{-\frac{2\theta}{(n+1)\sigma^2} x^{n+1}} dx + \frac{\sigma^2}{2} (\alpha + \beta e^{-\frac{2\theta}{(n+1)\sigma^2} b^{n+1}}) \right).$$

It is elementary to check that

$$J_1'(b) \to -\infty \ \text{ as } \ b \to 0^+, \quad \text{and} \quad J_1'(b) \to 0 \ \text{ as } \ b \to \infty.$$

Moreover, $J_1'(b)$ assumes the same sign as the term

$$\begin{aligned} \tilde{J}_1(b) \ &\equiv \ (b-1)^m - \theta\beta b^n - \frac{1}{\gamma(b)} \left(\int_0^b (x-\ell_0)^m e^{-\frac{2\theta}{(n+1)\sigma^2} x^{n+1}} dx + \frac{\sigma^2}{2} (\alpha + \beta e^{-\frac{2\theta}{(n+1)\sigma^2} b^{n+1}}) \right) \\ &\approx \ (b-1)^m - \theta\beta b^n - C \quad \text{as } b \to \infty, \end{aligned}$$

where

$$C = \frac{1}{\int_0^\infty e^{-\frac{2\theta}{(n+1)\sigma^2}x^{n+1}}dx}\left(\int_0^\infty (x-\ell_0)^m e^{-\frac{2\theta}{(n+1)\sigma^2}x^{n+1}}dx + \frac{\alpha\sigma^2}{2}\right).$$

We observe that whenever $m > n$ one has $\tilde{J}_1(b) \to \infty$ as $b \to \infty$. In this case the function $J_1(b)$ is decreasing for small b and increasing for large b. The minimum is then achieved for some finite value $b^\star$.

On the other hand, if $m \leq n$, then for some domain of "large values" of θ, β, α and σ, $\tilde{J}_1(b)$ can be negative for all b. In this case, the minimum is attained when $b \to \infty$. However, since $J_1'(b)$ goes to zero exponentially fast, so $J_1(b)$ will converge to its minimum value fast as $b \to \infty$. On the other hand, it is possible that for some "small values" of θ, β, α and σ, $J_1'(b)$ can be positive on some interval, and the minimum is attained when b is finite. By examining given set of model parameters, we can determine exactly which case is happening. Thus we conclude the following result.

Proposition 5.2. *Let $a = 0$, $h(x) = (x-\ell_0)^m$ and $f(x) = \theta x^n$ for some $m, n, \ell_0, \theta > 0$. (a) If $m > n$, then Problem 1 has a finite solution $b^\star \in (0,\infty)$. (b) If $m \leq n$, then there is a region of parameters (θ, β, α ,σ) such that the minimum is attained at $b^\star = \infty$; there is also a region of parameters (θ, β, α ,σ) such that the minimum is attained at finite $b^\star \in (0,\infty)$.*

In words, Proposition 5.2 provides the following economic insights: (a) Suppose the degree (level) m of cost $h(x)$, associated with an efficacy keeping the state close to the target level (i.e., the higher m implies the more costly efforts), exceeds the degree (strength) n of mean reverting drift $f(x)$ of the state process. In this situation, the decision maker can always find a non-trivial optimal price ceiling ($b^\star$) that is always finite. (b) Otherwise, the decision maker may suggest (depending on other model parameters) the "largest" possible price ceiling in the current market as a feasible solution to this optimization problem.

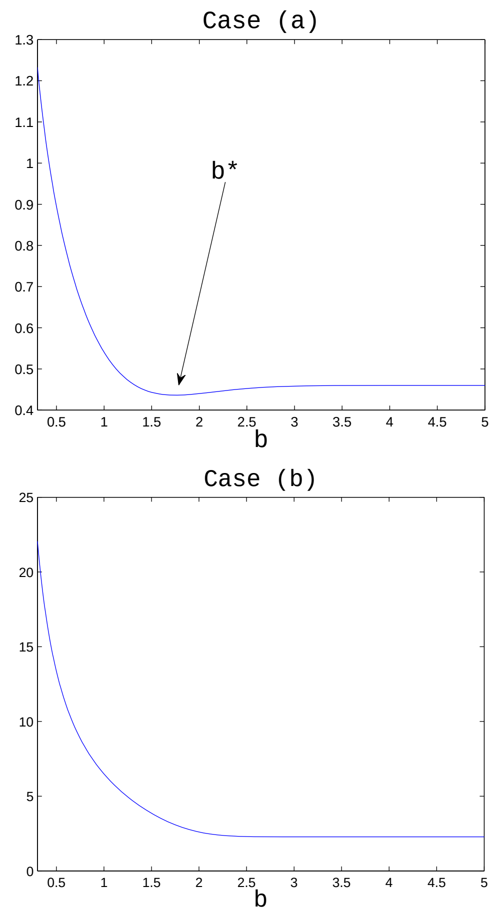

Figure 5.1: Plots of $J_1(b)$ in two situations
Upper panel: $m = 2$, $n = 1$, $\alpha = 0.2$, $\beta = 0.1$, $\theta = 0.7$, $\sigma^2 = 1$
Lower panel: $m = 2$, $n = 3$, $\alpha = 1.5$, $\beta = 1.7$, $\theta = 1$, $\sigma^2 = 4$

To locate the critical points of $J_1(b)$, one can use the Newton's method since we have a tractable second derivative of $J_1(b)$.

Numerical illustration

Figure 1 illustrates our results with examples. The left figure presents the case when $m = 2$, $n = 1$ (hence the case (a) in Proposition 5.2) and $\alpha = 0.2$, $\beta = 0.1$, $\theta = 0.7$, $\sigma^2 = 1$. The approximation of the optimal $b^\star$ can be obtained (with error less than

10^{-4}) by 1.7474. The right figure shows an example of a degenerate situation in the second case (b). With the model parameters given as $m = 2$, $n = 3$, $\alpha = 1.5$, $\beta = 1.7$, $\theta = 1$, and $\sigma^2 = 4$, one can see that the upper barrier $b = 2.5$ would already be a very good estimate for the optimal barrier $b^\star$.

The general case in considering $J_1(a,b)$ as a function of both a and b can be carried out in a similar way. For the sake of simplicity, we omit the details and present some numerical studies here. As before, $J_1(a,b)$ is clearly seen twice continuously differentiable. Based on all the related derivatives of $J_1(a,b)$, we still have two possible situations of (a) a unique finite solution or (b) a degenerate case. For the first case (a), in order to find the unique solution pair $(a^\star, b^\star)$ we can apply the 2-dimensional Newton's method by the following iterative relation:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix} - \mathbb{J}(a_n,b_n)^{-1} \begin{pmatrix} \frac{\partial J_1}{\partial a}(a_n,b_n) \\ \frac{\partial J_1}{\partial b}(a_n,b_n) \end{pmatrix},$$

where $\mathbb{J}(a_n,b_n)^{-1}$ is the inverse of the Jacobian matrix for $f_1 = \frac{\partial J_1}{\partial a}$ and $f_2 = \frac{\partial J_1}{\partial b}$. Figure 2 above presents an example when the parameters are given as $m = 2$, $n = 1$, $\theta = 1.5$, $\alpha = 0.01$, $\beta = 0.01$, and $\sigma = 10$. We can find a unique pair of optimal barriers $(a^\star, b^\star)$ that minimizes the ergodic cost $J_1(a,b)$. Numerical procedures are implemented using Maple and Matlab and the optimal barriers are given by $(a^\star, b^\star) \approx (0.0952, 1.9214)$ in the considered example.

5.3 Barrier controls over finite and infinite time horizon

We expand the previous ergodic problem to take into account more realistic application scenarios. More precisely, we consider barrier controls over finite and infinite time

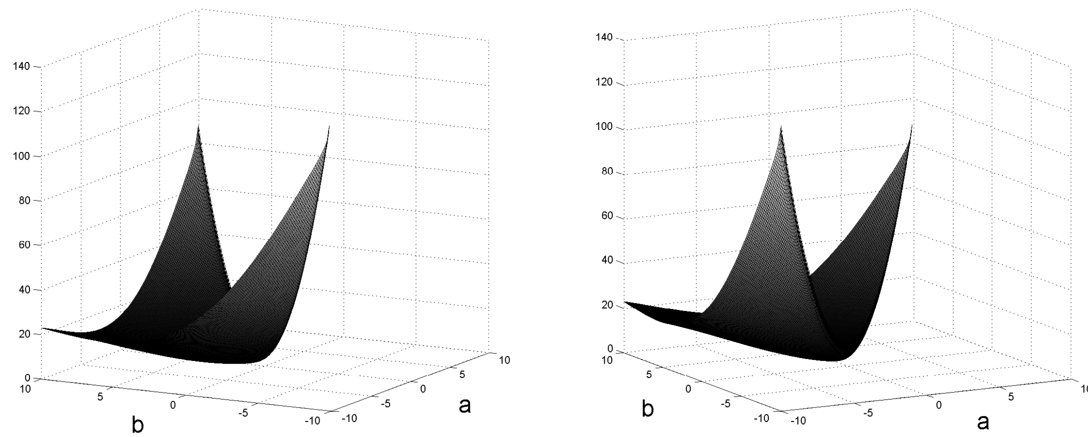

Figure 5.2: Plot of $J_1(a,b)$ from different view angles: Model parameters are $m=2$, $n=1$, $\theta=1.5$, $\alpha=0.01$, $\beta=0.01$, $\sigma=10$, and the minimizer $(a^{\star},b^{\star})$ is around $(0.0952, 1.9214)$.

horizon with discounted cost where a discount factor could vary from one period to another. For instance, we can incorporate the situation where interest rates are changing hourly, daily, or weekly basis. Also, our analysis allows a diffusion coefficient σ to be state-dependent, and henceforth we assume the diffusion coefficient $\sigma(x)$ is Lipschitz continuous on $[a,b]$. Moreover, for some technical reasons clarified later, we shall also assume f, $\sigma \in C_{+}^{2}([a,b])$, i.e., f and σ are twice continuously differentiable and positive on $[a,b]$. Our reflected SDE model becomes:

$$\begin{cases} dX_t = -f(X_t)dt + \sigma(X_t)dB_t + dL_t - dR_t, \\ X_0 = x \in [a,b], \end{cases} \tag{3.1}$$

In addition to the analysis steps used for Problem 1, we employ more sophisticated methods in identifying the solutions of the related Sturm-Liouville differential equations.

Problem 2 Our second control problem is to minimize the following utility functional for the process X_t in (3.1):

$$J_2(a,b) \equiv \sum_{k=0}^{\infty} \mathbb{E}\left\{\int_{t_k}^{t_{k+1}} e^{-\rho_k t} h_k(X_t)dt + \alpha_k \int_{t_k}^{t_{k+1}} e^{-\rho_k t} dL_t + \beta_k \int_{t_k}^{t_{k+1}} e^{-\rho_k t} dR_t\right\}, \tag{3.2}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < \infty$ are some given time epochs, and α_k, β_k represent the infinitesimal costs when the process X_t over the time interval $[t_k, t_{k+1})$ hits the barriers a and b, respectively. Also, $\rho_k \in (0,\infty)$ is an assumed discounted rate over the time period $[t_k, t_{k+1})$ and h_k is a given running cost function. The cost function $J_2(a,b)$ can also be considered over finite time horizon; the summation in (3.2) has finitely many summands and our computational schemes would cover such cases as well.

We proceed similarly as in Problem 1: Let $g_k(x)$ be a twice continuously differentiable function and applying Itô's formula to $e^{-\rho_k t} g_k(X_t)$ yields

$$\begin{aligned} e^{-\rho_k t} g_k(X_t) \quad &= \quad e^{-\rho_k t_k} g_k(X_{t_k}) + \int_{t_k}^{t} \Big[e^{-\rho_k s}[L-\rho_k] g_k(X_s) ds + e^{-\rho_k s} g_k'(X_s) dL_s \\ &\qquad - e^{-\rho_k s} g_k'(X_s) dR_s \Big] + \sigma \int_{t_k}^{t} e^{-\rho_k s} g_k'(X_s) dB_s\,, \end{aligned}$$

where $t_k \le t \le t_{k+1}$ and L is the following operator:

$$Lg(x) = \frac{1}{2}\sigma^2(x) g''(x) - f(x) g'(x)\,.$$

If g_k satisfies the following second order differential equation with boundary conditions

$$(L-\rho_k) g_k(x) = h_k(x) \quad \text{for } a < x < b\,, \quad g_k'(a) = \alpha_k \quad \text{and} \quad g_k'(b) = -\beta_k\,, \tag{3.3}$$

then

$$\int_{t_k}^{t_{k+1}} e^{-\rho_k t} h_k(X_t)dt + \alpha_k \mathbb{E} \int_{t_k}^{t_{k+1}} e^{-\rho_k t} dL_t + \beta_k \mathbb{E} \int_{t_k}^{t_{k+1}} e^{-\rho_k t} dR_t$$
$$= e^{-\rho_k t_{k+1}} \mathbb{E}\, g_k(X_{t_{k+1}}) - e^{-\rho_k t_k} \mathbb{E}\, g_k(X_{t_k}) .$$

Hence, we have

$$J_2(a,b) = \sum_{k=0}^{\infty} \left\{ e^{-\rho_k t_{k+1}} \mathbb{E}\, g_k(X_{t_{k+1}}) - e^{-\rho_k t_k} \mathbb{E}\, g_k(X_{t_k}) \right\} . \tag{3.4}$$

In what follows we shall assume $t_k = k\Delta$ for some $\Delta > 0$, that is, the time period is equidistant. In this case, the Δ-skeleton $\{X_{t_0}, X_{t_1}, \cdots, X_{t_k}, \cdots\}$ of the process $\{X_t\}$ forms a Markov chain. Notice that if we can derive a tractable form of the transition probability density of X_t, then the optimization of the above expectation (3.4) would become solvable analytically or numerically. It is well-known that the transition probability density $p_t(x,y)$ of the process X_t is given by the following PDE

$$\frac{\partial}{\partial t} p_t(x,y) = \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} p_t(x,y) - f(y) \frac{\partial}{\partial y} p_t(x,y) , \tag{3.5}$$

with boundary conditions

$$\lim_{t \to 0} p_t(x,y) = \delta(y-x) , \quad \frac{\partial}{\partial y} p_t(x,a) = \frac{\partial}{\partial y} p_t(x,b) = 0 , \tag{3.6}$$

where $\delta(\cdot)$ denotes the Dirac delta function. The above equations (3.5)–(3.6) can be expressed by the following spectral expansion:

$$p_t(x,y) = m(y) \sum_{n=0}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y), \quad t > 0 , \tag{3.7}$$

where λ_n and $\varphi_n(x)$ are the eigenvalues and normalized eigenfunctions (by “normalized" we mean that $\int_{-\infty}^{\infty}\varphi_n^2(x)dx = 1$), respectively, and the pair $(\lambda_n, \varphi_n(x))$ is defined via the following Sturm-Liouville (SL) equation

$$-Lu(x) = \lambda u(x) \tag{3.8}$$

with boundary conditions

$$u'(a) = 0, \qquad u'(b) = 0. \tag{3.9}$$

The problem is thus reduced to determining the eigenvalues λ_n and the corresponding solutions $u(x, \lambda_n)$ for the SL equation (3.8). In the Appendix, we provide rigorous theoretical procedures towards finding the optimal barriers for Problem 2, which are based on a general solution of the SL equation with boundary conditions; see Kravchenko and Porter [17] for more details.

Those solution steps described in the Appendix are theoretically well justified, however, the associated computational load can be quite intensive in practice. Instead, to simplify the computations, we propose an alternative approach based on the eigenvalue and eigenfunction asymptotics by Linetsky [20], which was originated from Fulton and Pruess [23]. To assure the validity of conversion to Liouville Normal form in the aforementioned paper, we shall require the assumption that f, $\sigma \in C_+^2([a,b])$ (twice continuously differentiable and positive on $[a,b]$). Below is a summary of results adapted to our model using the procedures in [23].

We start with a change of variable:

$$y(x) = \int_a^x \sqrt{m(r)s(r)}dr = \sqrt{2}\int_a^x \frac{1}{\sigma(r)}dr.$$

Let $B = y(b)$. Define

$$F(y) = \frac{2^{\frac{1}{4}}}{\sqrt{\sigma(x(y))s(x(y))}}, \quad \text{and} \quad Q(y) = \frac{F''(y)}{F(y)},$$

where $x(y)$ denotes the inverse function of $y(x)$. Then we introduce the following notations:

$$W(x) = 2^{-\frac{3}{2}} \left(\sigma'(x) + \frac{2f(x)}{\sigma(x)} \right),$$
$$A(x) = W(a) + \frac{1}{\sqrt{2}} \int_a^x \frac{Q(y(r))}{\sigma(r)} dr,$$
$$C = W(a) + W(b) + \frac{1}{2} \int_0^B Q(r) dr.$$

Now the eigenvalue and eigenfunction asymptotics for the transition density $p_t(x,y)$ are as follows:

$$\lambda_n = \frac{n^2\pi^2}{B^2} + A_0 + \frac{C^2}{n^2\pi^2} + O(\frac{1}{n^4}), \tag{3.10}$$

$$\varphi_n(x) = \pm 2^{\frac{1}{4}} \sqrt{\frac{\sigma(x)s(x)}{B}} \left\{ \cos\left(\frac{n\pi y(x)}{B}\right) + \frac{1}{n\pi} (BA(x) - Cy(x)) \sin\left(\frac{n\pi y(x)}{B}\right) \right\} + O(\frac{1}{n^2}) \tag{3.11}$$

where $A_0 = \frac{2C}{B}$ and $\varphi_n(x)$'s are defined up to an overall sign.

Recalling the spectral expansion of transition probability density in (3.7), and based on the above terms, we can obtain an explicit expression of $p_t(x,y)$:

$$p_t(x,y) = \pi(y) + m(y) \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y), \quad t > 0. \tag{3.12}$$

Notice that the first term in the expansion corresponds to eigenvalue $\lambda_0 = 1$ and thus the stationary density π.

Numerical illustration

In the remaining part of this section we illustrate our results by a numerical study. We consider the case when there are only two terms in the summation of (3.2) with $t_0 = 0$, $t_1 = 1$, and $t_2 = 2$. In this case, we have

$$\begin{aligned} J_2(a,b) &= \sum_{k=0}^{1} \left\{ e^{-\rho_k t_{k+1}} \mathbb{E}\, g_k(X_{t_{k+1}}) - e^{-\rho_k t_k} \mathbb{E}\, g_k(X_{t_k}) \right\} \\ &= e^{-\rho_0} \mathbb{E}\, g_0(X_1) - g_0(X_0) + e^{-2\rho_1} \mathbb{E}\, g_1(X_2) - e^{-\rho_1} \mathbb{E}\, g_1(X_1)\,. \end{aligned} \tag{3.13}$$

We note that the discounted cost expression of $J_2(a,b)$ (as well as $J_2(b)$ below) is much more complicated than those for Problem 1, because it involves the transition probability density as well as g_i, $i = 1,2$. As a consequence, it is onerous for one to directly verify the interactions among all the model parameters and their influences to the sign of derivatives.

We shall consider the following setting for the sake of simplicity of numerical experiment: $f(x) = x$, $X_0 = 0.5$, $a = 0$, $\sigma = \sqrt{2}$, $\rho_0 = \rho_1 = 1$, $h_0(x) = x$ and $h_1(x) = x^2$. For an arbitrary time interval $[t_k, t_{k+1}]$, the equation (3.3) is

$$\begin{cases} \dfrac{1}{2}\sigma^2 g''(x) - f(x)g'(x) - \rho g(x) = h(x), \quad a < x < b\,, \\ g'(a) = \alpha \quad \text{and} \quad g'(b) = -\beta\,, \end{cases} \tag{3.14}$$

The solution is of the form:

$$g(x) = g^*(x) + \hat{g}(x), \tag{3.15}$$

where $g^*(x)$ is one particular solution of the ordinary differential equation in (3.14), and $\hat{g}(x)$ is the general solution of the homogeneous form of the equation (3.14), and it is given by

$$\hat{g}(x) = \phi(x)\left(C_1 + C_2\int_a^x \frac{1}{\phi^2(u)}s(u)du\right).$$

Here, $\phi(x)$ is one non-zero solution of the homogeneous form of the ordinary differential equation in (3.14), and also recall that

$$s(x) = \exp\left(\frac{2}{\sigma^2}\int_a^x f(r)dr\right).$$

Based on the previous setting of model parameters, we have:

$$\begin{aligned}
&\phi(x) = e^{\frac{x^2}{2}},\\
&\hat{g}_0(x) = e^{\frac{x^2}{2}}\left(C_1 + C_2\int_0^x e^{-r^2+r}dr\right), \quad \text{and} \quad \hat{g}_1(x) = e^{\frac{x^2}{2}}\left(C_1' + C_2'\int_0^x e^{-r^2+r}dr\right),\\
&g_0^*(x) = -\frac{1}{2}x, \quad \text{and} \quad g_1^*(x) = -\frac{1}{3}x^2 - \frac{2}{3}.
\end{aligned}$$

Thus by (3.15),

$$\begin{aligned}
g_0(x) &= e^{\frac{x^2}{2}}\left(C_1 + C_2\int_0^x e^{-r^2+r}dr\right) - \frac{1}{2}x,\\
g_1(x) &= e^{\frac{x^2}{2}}\left(C_1' + C_2'\int_0^x e^{-r^2+r}dr\right) - \frac{1}{3}x^2 - \frac{2}{3}.
\end{aligned}$$

To determine the above constants C_i, C_i' $(i = 1,2)$, we take $\alpha_0 = \beta_0 = \frac{1}{2}$, and $\alpha_1 = \beta_1 = \frac{1}{3}$. Therefore, we can finally get g_0 and g_1 as follows:

$$\begin{aligned}
g_0(x) &= -\frac{1}{2}x - e^{\frac{x^2}{2}}\left(\frac{1}{b}e^{-b^2+b} + \int_x^b e^{-r^2+r}dr\right),\\
g_1(x) &= -\frac{1}{3}x^2 - \frac{2}{3} + \frac{1}{3}e^{\frac{x^2}{2}}\left(e^{-\frac{b^2}{2}}(2-\frac{1}{b}) - \frac{1}{b}e^{-b^2+b} - \int_x^b e^{-r^2+r}dr\right).
\end{aligned} \tag{3.16}$$

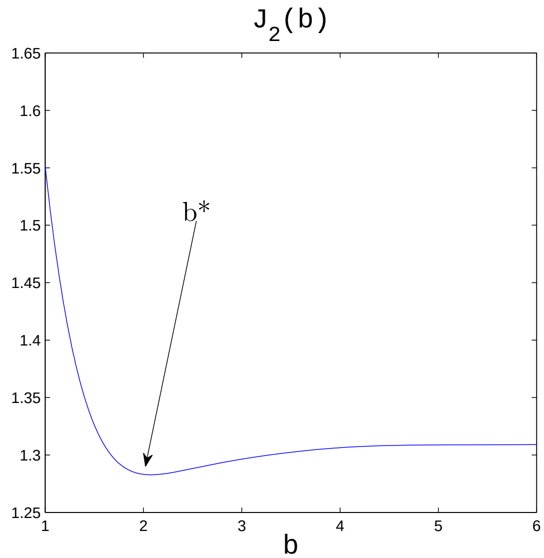

Figure 5.3: Plot of $J_2(b)$: Model parameters are given as $f(x) = x$, $X_0 = 0.5$, $a = 0$, $\sigma = \sqrt{2}$, $\alpha_0 = \beta_0 = \frac{1}{2}$, and $\alpha_1 = \beta_1 = \frac{1}{3}$. The optimal barrier is found to be around $b^\star \approx 2.0779$.

Thus by the above results and (3.12) we obtain the following expression of (3.13):

$$\begin{aligned} J_2(b) &= e^{-1}\mathbb{E}\, g_0(X_1) - g_0(X_0) + e^{-2}\mathbb{E}\, g_1(X_2) - e^{-1}\mathbb{E}\, g_1(X_1) \\ &= e^{-1}\int_0^b (g_0(y) - g_1(y))\, p_1(0.5, y)dy - g_0(0.5) + e^{-2}\int_0^b g_1(y)p_2(0.5, y)dy. \end{aligned} \tag{3.17}$$

Since the spectral expansion of transition probability (3.12) decays exponentially fast as $\lambda_n \to \infty$, we can use finite sum to approximate $p_1(0.5, y)$ and $p_2(0.5, y)$ in (3.17). We shall use the first 3 terms for numerical approximation. (See Remark 5.3 below.) Using the eigenvalue expression in (3.10), we get:

$$\begin{aligned} \lambda_1 &\approx \frac{\pi^2}{b^2} + (\frac{1}{2} + \frac{b^2}{12}) + \frac{b^2(6+b^2)^2}{576\pi^2}, \\ \lambda_2 &\approx \frac{4\pi^2}{b^2} + (\frac{1}{2} + \frac{b^2}{12}) + \frac{b^2(6+b^2)^2}{2304\pi^2}, \\ \lambda_3 &\approx \frac{9\pi^2}{b^2} + (\frac{1}{2} + \frac{b^2}{12}) + \frac{b^2(6+b^2)^2}{5184\pi^2}. \end{aligned} \tag{3.18}$$

Also by (3.11) we can get

$$\varphi_n(x) \approx \frac{2^{\frac{3}{4}} e^{\frac{x^2}{4}}}{b^{\frac{1}{2}}} \left(\cos(\frac{n\pi x}{b}) + \frac{b}{24n\pi}(x(x^2-6)-6-b^2)\sin(\frac{n\pi x}{b}) \right). \tag{3.19}$$

Now we employ the Newton's method again. With the help of Maple and Matlab, we conclude that the optimal barrier is given as $b^\star \approx 2.0779$ in the considered example and the graph of $J_2(b)$ is displayed in Figure 3.

Remark 5.3. *To justify the 3-term approximation for the spectral expansion of transition probability, we display the numerical results using $n = 3$ terms and $n = 8$ terms in Figure 4. Notice that the curves of $J_2(b)$ are almost identical and therefore incorporating additional higher order terms does not affect the sought-after value $b^\star$ in any significant way. The error analysis in general case seems to be involved and also beyond the scope/focus of the current paper. We refer the reader to [14], which provides the detailed analysis on the computational issues (cf. Section 4 therein) for the spectral expansion of the hitting time density for the reflected Brownian motions.*

5.4 Concluding remarks

We have considered the control problems of finding *optimal* pricing barriers under a *regulated* financial market system using a class of one-dimensional reflected SDEs. For instance, the prices of basic goods and services, as well as the foreign exchange rates and the domestic interest rates could undergo macro-interventions by the government or other third part authority, and as a result, the resulting price dynamics are controlled by the price ceiling and the price floor. By analyzing the associated reflected SDE models, we derive explicit expressions of the cost functionals related with a long time

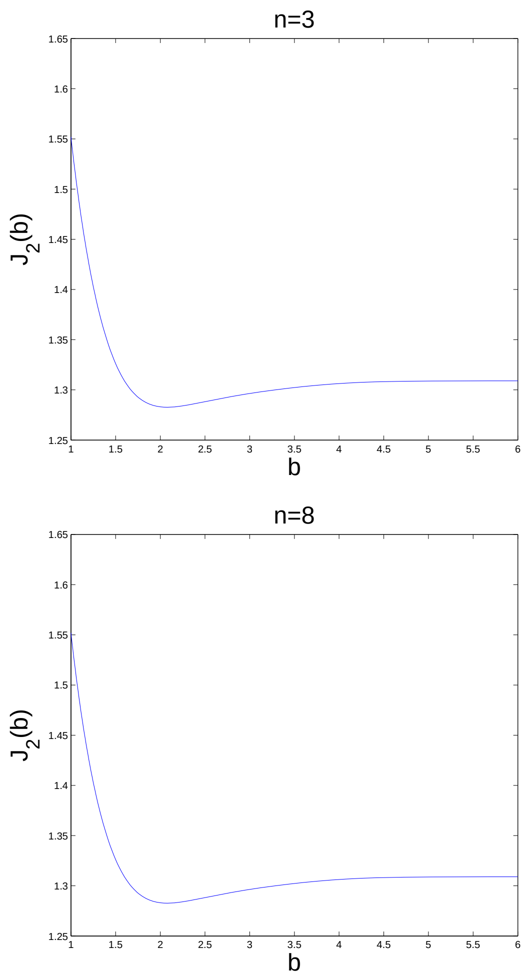

Figure 5.4: $J_2(b)$ using up to $n = 3$ terms in the spectral expansion (upper panel) and using $n = 8$ terms (lower panel). The (maximum) difference of $J_2(b)$ between the two cases is negligible; it is of the order 10^{-6}. The resulting $b^{\star}$ values are indeed very close to each other; they are 2.0779 and 2.0800 for $n = 3$ and $n = 8$, respectively.

average cost and a finite or infinite time horizon discount cost where the discount factor is allowed to vary from one period to another. We illustrate our results of quantifying the set of optimal pricing barriers via concrete numerical studies. In order to expand the applicability of our methods, we are currently working on the reflected SDE model with jumps (cf. [4]). A practical implication of such models is that the asset price or interest rates, under the regulated financial market, are allowed to change according to jump size distribution (as reactions to, e.g., outside good/bad news), and thus it can capture the more realistic empirical market characteristics.

Appendix

A. Proof of Proposition 5.1

We begin by computing $\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}\, h(X_t)dt$. To identify this limit we define $s(x)$ and $m(x)$ as the scale and the speed densities:

$$s(x) = \exp\left(\frac{2}{\sigma^2}\int_a^x f(r)dr\right), \quad \text{and} \quad m(x) = \frac{2}{\sigma^2 s(x)}. \tag{A1}$$

First, we follow the approach in [16] to derive the invariant measure of this diffusion process X_t. If the stationary density $\psi(x)$ exists, then it satisfies the Kolmogorov backward equation

$$\frac{\sigma^2}{2}\frac{d^2}{dx^2}(\psi(x)) + \frac{d}{dx}(f(x)\psi(x)) = 0.$$

Integrating the above equation, and then multiplying $s(x)$ on both sides, we obtain the following differential equation.

$$\frac{d}{dx}(s(x)\psi(x)) = C_1\frac{2}{\sigma^2}s(x),$$

where C_1 is a constant. Integrating this equation again yields

$$\psi(x) = \frac{2}{\sigma^2 s(x)} \left(C_1 \int_a^x s(y)dy + C_2 \right) = m(x) \left(C_1 \int_a^x s(y)dy + C_2 \right),$$

for some constants C_1 and C_2. If we can find the constants C_1 and C_2 such that $\psi(x)$ is a probability density function on $[a,b]$, then a stationary density exists. To this end, we can choose $C_1 = 0$, and then $C_2 = (\int_a^b m(x)dx)^{-1} \in (0,\infty)$. If we define

$$\pi_{a,b}(x) \equiv m(x) \left(\int_a^b m(x)dx \right)^{-1} = \frac{e^{-\frac{2}{\sigma^2}\int_a^x f(y)dy}}{\int_a^b e^{-\frac{2}{\sigma^2}\int_a^x f(y)dy} dx}, \tag{A2}$$

then the process $\{X_t\}$ has invariant measure $\pi_{a,b}(x)$. To prove the (V-uniform) ergodicity of this process, it suffices to check that $\{X_t\}$ satisfies Condition 5.5 in [8], i.e., there exists a bounded set $A \subset [a,b]$ and $\delta > 0$, such that for all $x \in [a,b] \cap A^c$, $f(x) \geq \delta$. This is self-evident since $f : \mathbb{R} \to \mathbb{R}_+$ is Lipschitz continuous and not identically zero on $[a,b]$. Thus we can conclude that $\{X_t\}$ is ergodic, which implies

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T h(X_t)dt = \int_a^b h(x)\pi_{a,b}(x)dx \quad a.s.$$

Since h is bounded and integrable on $[a,b]$, by the dominated convergence theorem we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}\, h(X_t)dt = \int_a^b h(x)\pi_{a,b}(x)dx. \tag{A3}$$

Now we compute $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\ [\alpha L_T + \beta R_T]$. Consider a twice continuously differentiable function $g(x)$. An Itô's formula yields (see [13] for details about Itô formula for reflected process):

$$g(X_T) = g(x) + \int_0^T \left[Lg(X_t)dt + g'(X_t)dL_t - g'(X_t)dR_t \right] + \sigma \int_0^T g'(X_t)dB_t\,,$$

where

$$Lg(x) = \frac{\sigma^2}{2} g''(x) - f(x)g'(x)\,.$$

If g satisfies

$$Lg(x) = -\lambda \quad \text{for } \; a < x < b\,, \quad g'(a) = \alpha \quad \text{and} \quad g'(b) = -\beta\,, \tag{A4}$$

then

$$\mathbb{E}\, g(X_T) = g(x) - \lambda T + \alpha \mathbb{E}\, L_T + \beta \mathbb{E}\, R_T\,.$$

This means

$$\frac{1}{T}\left[\alpha \mathbb{E}\, L_T + \beta \mathbb{E}\, R_T\right] = \lambda + \frac{1}{T}\mathbb{E}\, g(X_T) - \frac{1}{T} g(x)\,.$$

Since X_T is always in the interval $[a,b]$ and g is continuous on $[a,b]$, hence bounded on $[a,b]$. Thus, we have

$$\lim_{T\to\infty} \frac{1}{T}\left[\alpha \mathbb{E}\, L_T + \beta \mathbb{E}\, R_T\right] = \lambda\,. \tag{A5}$$

Therefore, to find $\lim_{T\to\infty} \frac{1}{T}\left[\alpha \mathbb{E}\, L_T + \beta \mathbb{E}\, R_T\right]$, it suffices to find the constant λ which is determined by (A4). To solve (A4), we write

$$Lg(x) = \frac{\sigma^2}{2} g''(x) - f(x)g'(x) = \frac{1}{m(x)}\left(\frac{g'(x)}{s(x)}\right)'. \tag{A6}$$

Thus the equation (A4) implies $\frac{1}{m(x)}\left(\frac{g'(x)}{s(x)}\right)' = -\lambda$, which in turn yields for some $C > 0$ that

$$\frac{g'(x)}{s(x)} = -\lambda \int_a^x m(u)du + C\,.$$

Noticing $s(a) = 1$, the condition $g'(a) = \alpha$ implies $C = \alpha$. Also, the condition $g'(b) = -\beta$ becomes

$$-\beta = s(b)\left[\alpha - \lambda \int_a^b m(u)du\right].$$

Therefore, we obtain that

$$\lambda = \frac{\alpha s(b) + \beta}{s(b) \int_a^b m(u) du}.$$

By the definitions of $s(x)$ and $m(x)$, we have

$$\lambda = \frac{\sigma^2}{2} \left(\alpha + \beta e^{-\frac{2}{\sigma^2} \int_a^b f(r) dr} \right) \left(\int_a^b e^{-\frac{2}{\sigma^2} \int_a^u f(r) dr} du \right)^{-1}.$$

Combining this with (A5) we have

$$\lim_{T \to \infty} \frac{1}{T} [\alpha \mathbb{E}\, L_T + \beta \mathbb{E}\, R_T] = \frac{\sigma^2}{2} \left(\alpha + \beta e^{-\frac{2}{\sigma^2} \int_a^b f(r) dr} \right) \left(\int_a^b e^{-\frac{2}{\sigma^2} \int_a^u f(r) dr} du \right)^{-1}.$$

Together with (A3), we obtain the desired result in (2.2). ■

B. Theoretical procedures towards solving Problem 2

Recalling the definitions of $s(x)$, $m(x)$ in (A1), now we have

$$s(x) = \exp\left(2 \int_a^x \frac{f(r)}{\sigma^2(r)} dr \right), \quad \text{and} \quad m(x) = \frac{2}{\sigma^2(x) s(x)}. \tag{A7}$$

Also, recalling the equation (A6), we see that the equation (3.8) is now equivalent to

$$(-s(x)^{-1} u'(x))' = \lambda m(x) u(x). \tag{A8}$$

Fortunately, a recent study by Kravchenko and Porter [17] provides a general solution of the SL equation of the form in (A8) with boundary condition (3.9), in terms of a known non-trivial solution u_0 of the equation

$$(-s(x)^{-1} u_0'(x))' = 0.$$

In what follows, we shall adopt the approach in Kravchenko and Porter [17]. First we choose, for $x \in [a,b]$,

$$u_0(x) = \int_a^x e^{2\int_a^z \frac{f(r)}{\sigma^2(r)}dr} dz + 1. \tag{A9}$$

Then, the general solution of (A8) has the form $u = c_1 u_1 + c_2 u_2$ where c_1, c_2 are arbitrary complex numbers and the functions u_i $(i = 1,2)$ are given by

$$u_1 = u_0 \sum_{k=0}^{\infty} \lambda^k \tilde{X}^{(2k)} \quad \text{and} \quad u_2 = u_0 \sum_{k=0}^{\infty} \lambda^k X^{(2k+1)}. \tag{A10}$$

Here, $\tilde{X}^{(n)}$ and $X^{(n)}$ are defined by the following recursive relations: $\tilde{X}^{(0)} = X^{(0)} = 1$,

$$\tilde{X}^{(n)}(x) = \begin{cases} \int_a^x \tilde{X}^{(n-1)}(r) u_0^2(r) m(r) dr, & \text{if } n \text{ is odd} \\ \\ -\int_a^x \tilde{X}^{(n-1)}(r) \frac{s(r)}{u_0^2(r)} dr, & \text{if } n \text{ is even} \end{cases} \tag{A11}$$

and

$$X^{(n)}(x) = \begin{cases} -\int_a^x X^{(n-1)}(r) \frac{s(r)}{u_0^2(r)} dr, & \text{if } n \text{ is odd} \\ \\ \int_a^x X^{(n-1)}(r) u_0^2(r) m(r) dr, & \text{if } n \text{ is even.} \end{cases} \tag{A12}$$

The boundary conditions (3.9) and the normalizing condition can be used to determine the coefficients λ, c_1 and c_2. From the definitions of u_1 and u_2, we have

$$\begin{cases} u_1(a) = u_0(a), u_2(a) = 0, \\ \\ u_1'(a) = u_0'(a), u_2'(a) = -s(a)/u_0(a) \end{cases} \tag{A13}$$

and

$$u_1' = \frac{u_0'}{u_0}u_1 - \frac{s}{u_0}\sum_{k=1}^{\infty}\lambda^k \tilde{X}^{(2k-1)}, \quad u_2' = \frac{u_0'}{u_0}u_2 - \frac{s}{u_0}\sum_{k=0}^{\infty}\lambda^k X^{(2k)}. \tag{A14}$$

Using the relations in (A13), and the fact that $u_0'(x) = s(x)$, we can write the boundary condition $u'(a) = 0$ as

$$c_1 - c_2\frac{1}{u_0(a)} = 0 \quad \text{or} \quad c_2 = c_1 u_0(a)\,.$$

Using (A14) we can rewrite $u'(b) = 0$ as

$$c_1\left(u_0'(b)u_1(b) - s(b)\sum_{k=1}^{\infty}\lambda^k\tilde{X}^{(2k-1)}(b)\right) + c_2\left(u_0'(b)u_2(b) - s(b)\sum_{k=0}^{\infty}\lambda^k X^{(2k)}(b)\right) = 0\,,$$

which can be simplified, from noting $c_2 = c_1u_0(a)$ and $u_0'(b) = s(b)$, as

$$u_0(b)\sum_{k=0}^{\infty}\lambda^k\tilde{X}^{(2k)}(b) - \sum_{k=1}^{\infty}\lambda^k\tilde{X}^{(2k-1)}(b) + u_0(a)\left(u_0(b)\sum_{k=0}^{\infty}\lambda^k X^{(2k+1)}(b) - \sum_{k=0}^{\infty}\lambda^k X^{(2k)}(b)\right) = 0. \tag{A15}$$

Notice that

$$X^{(1)}(b) = -\int_a^b \frac{s(r)}{u_0^2(r)}dr = -\int_a^b \frac{1}{u_0^2(r)}du_0(r) = \frac{1}{u_0(b)} - \frac{1}{u_0(a)}.$$

The constant coefficients of the series of λ, that is, when $k = 0$ in (A15), satisfy

$$u_0(b) + u_0(a)u_0(b)X^{(1)}(b) - u_0(a) = 0.$$

As a result, our problem now reduces to finding zeros of the analytic function

$$\phi(\lambda) = \sum_{n=1}^{\infty} a_n\lambda^n\,, \tag{A16}$$

where

$$a_n = u_0(b)\tilde{X}^{(2n)}(b) - \tilde{X}^{(2n-1)}(b) + u_0(a)\left(u_0(b)X^{(2n+1)}(b) - X^{(2n)}(b)\right). \qquad \text{(A17)}$$

Thus, we are able to outline theoretical procedures towards finding the optimal barriers for Problem 2:

1. First, compute $u_0(x)$ as in (A9).

2. Construct functions $\tilde{X}^{(n)}$ and $X^{(n)}$ according to (A11) and (A12), respectively.

3. Find λ_n by computing the zeros of the function $\phi(\lambda)$ defined in (A16).

4. Obtain $p_t(x,y)$ by (3.7) with λ_n and $\varphi_n(x) = c_1\left(u_1(x) + u_0(a)u_2(x)\right)$, where u_1 and u_2 are defined as in (A10).

5. Compute $J_2(a,b)$ in (3.4) by the Markov chain expression using the associated transition probability density function of X_{t_k} given by $p_{t_k}(x,y)$.

6. Finally, minimize $J_2(a,b)$.

We note that while the above steps are theoretically solid, once putting into practice, one encounters some unavoidable issues. For instance, Step 3 above requires for us to use formula (A10) which is an infinite series. We were unable to find theoretical (or empirical) principles about how to truncate such infinite series for numerical purposes.

Chapter 6

Optimal barrier for one type of reflected regime-switching process

6.1 Introduction

Suppose that $\{J_t, t \geq 0\}$ is an irreducible and recurrent continuous-time Markov chain with finite state space $\mathscr{S}$. Consider the Markov-modulated reflected diffusion process $\{X_t, t \geq 0\}$ which is the solution of the following stochastic differential equation:

$$\begin{cases} dX_t = b(J_t, X_t)dt + \sigma(J_t, X_t)dB_t + dL_t - dR_t \quad t \geq 0\,, \\ X_0 = x \in [k, K]\,, \end{cases} \tag{A1}$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion, k and K are real numbers, b and σ are some given functions from $\mathscr{S} \times \mathbb{R} \to \mathbb{R}$, $\{L_t, t \geq 0\}$ is the minimal nondecreasing process that keeps $X_t \geq k$ and increases only when X_t hits k, and $\{R_t, t \geq 0\}$ is the minimal nondecreasing process that keeps $X_t \leq K$ and increases only when X_t hits K.

Plenty of literature in Economics has shown strong evidence for the existence of regime changes. For one instance, in history unusually high volatility of the U.S. short-term interest rates had occurred during the episodes of the 1973 and 1979 OPEC oil crises, the 1979-82 Federal Reserve Monetary Experiment, and the 1987 stock market crash. For more supporting materials, we refer the reader to [10] and the references therein. Thus the Markov chain $\{J_t, t \geq 0\}$ can be used to represent the economic environment. For example, if J_t only attains two states, then state 1 and 2 can stand for 'Bull' and 'Bear' stock markets, or 'Low' and 'High' volatility states. and X_t can be used to represent some economic quantity. We are interested in finding the optimal barriers k and K to maximize utility functions of ergodic type. For example, we are interested in finding the pair of optimal k and K to optimize $\alpha \mathbb{E}\, h(X_\infty) - \beta f(\mathbf{Var}(X_\infty))$ or $\dfrac{\mathbb{E}\, h(X_\infty)}{f(\mathbf{Var}(X_\infty))}$. We refer to [12] for a motivation of the problem in the absence of the Markov modulated process J_t.

To obtain explicit solutions of the problems, we assume $K = \infty$ and we limit ourselves to the case that the $\mathscr{S}$ contains only two states, i.e., $\mathscr{S} = \{1, 2\}$. The infinitesimal generator matrix of the Markov chain $\{J_t, t \geq 0\}$ is

$$Q := \begin{pmatrix} -\mu_1 & \mu_1 \\ \mu_2 & -\mu_2 \end{pmatrix}.$$

We also assume that the process $\{X_t, t \geq 0\}$ satisfies the following reflected Langevin equation:

$$\begin{cases} dX_t = -(b(J_t) + \lambda(J_t)X_t)dt + \sigma(J_t)dB_t + dL_t, \quad t \geq 0, \\ X_0 = x \in [k, \infty], \end{cases} \tag{A2}$$

where

(i) σ, λ and b are strictly positive functions defined on $\mathscr{S}$.

(ii) k is a given positive real number;

(iii) $\{B_t, t \geq 0\}$ is a standard Brownian motion independent of J_t;

(iv) $\{L_t, t \geq 0\}$ is the minimal nondecreasing process that keeps $X_t \geq k$ and increases only when X_t hits k. Namely, $\{L_t, t \geq 0\}$ is the local time.

That is to say, we consider only the Markov modulated reflected Ornstein-Uhlenbeck process. We want to find an optimal k to maximize the utility $\alpha \mathbb{E}\, h(X_\infty) - \beta f(Var(X_\infty))$ or $\dfrac{\mathbb{E}\, h(X_\infty)}{f(\mathbf{Var}(X_\infty))}$. We will use a result of [28].

6.2 Moments of the invariant measures

Let $Y_t = X_t - k$. Then Y_t satisfies the following reflected Langevin equation:

$$dY_t = -(b(J_t) + \lambda(J_t)(Y_t + k))dt + \sigma(J_t)dB_t + d\tilde{L}_t, \tag{A1}$$

where $\tilde{L}_t$ is the local time of Y_t, namely, $\{\tilde{L}_t, t \geq 0\}$ is the minimal nondecreasing process that keeps $Y_t \geq 0$ and increases only when Y_t hits 0. From now on for simplicity we denote $\sigma_i = \sigma(i)$, $\lambda_i = \lambda(i)$ and $b_i = b(i)$.

From [28, Theorem 3.1 and Theorem 3.2] we see that the stationary density of (Y_t, J_t) exists. Namely, there are two probability density functions $\mathscr{G}_1(x)$ and $\mathscr{G}_2(x)$ such that for $i = 1, 2$,

$$P(Y_\infty \leq x, J_\infty = i) = \int_0^x \mathscr{G}_i(u)du, \quad \forall\, x \geq 0.$$

Moreover, if $\hat{g}_i(\alpha) = \int_0^\infty e^{-\alpha x}\mathscr{G}_i(x)dx$ is the Laplace transform of $\mathscr{G}_i$, then $\hat{g}(\alpha) = (\hat{g}_1(\alpha), \hat{g}_2(\alpha))^T$ satisfies

$$A(\alpha)\hat{g}(\alpha) - B(\alpha)\hat{g}'(\alpha) = \alpha P, \tag{A2}$$

where

$$\begin{aligned}
A(\alpha) &= Q^T + \mathrm{diag}(\sigma_i^2\alpha^2/2 + (b_i + k\lambda_i)\alpha) \\
&= \begin{pmatrix} -\mu_1 + \frac{\sigma_1^2}{2}\alpha^2 + (b_1 + k\lambda_1)\alpha & \mu_2 \\ \mu_1 & -\mu_2 + \frac{\sigma_2^2}{2}\alpha^2 + (b_2 + k\lambda_2)\alpha \end{pmatrix} \\
B(\alpha) &= \mathrm{diag}(\lambda_i\alpha) \\
P &= (p_1, p_2),
\end{aligned} \tag{A3}$$

and p_1 and p_2 are two constants independent of α such that

$$p_1 + p_2 = \frac{\lambda_1\mu_2 + \lambda_2\mu_1}{\mu_1 + \mu_2}\left(\int_0^\infty e^{-(\frac{n}{4}s^2 + (m+k)s)}ds\right)^{-1}.$$

Now we are going to solve Equation (A2) for some special cases. A solvable situation which is more generalized than the examples provided in [28] is when the parameter set satisfies $b_i = m\lambda_i$, $\sigma_i = \sqrt{n\lambda_i}$ $(m, n > 0)$ for all i. To explicitly solve this case, first let's define

$$h(\alpha) = \lambda_1\hat{g}_1(\alpha) + \lambda_2\hat{g}_2(\alpha).$$

Adding the two equations in (A2) together one can get

$$(\frac{n}{2}\alpha + m + k)h(\alpha) - h'(\alpha) = p_1 + p_2. \tag{A4}$$

Let $y(\alpha) = e^{-(\frac{n}{4}\alpha^2+(m+k)\alpha)}h(\alpha)$. Then the above differential equation becomes

$$y'(\alpha) = -e^{-(\frac{n}{4}\alpha^2+(m+k)\alpha)}(p_1+p_2).$$

Integrating both sides from 0 to α we obtain $y(\alpha)$ and thus

$$h(\alpha) = e^{\frac{n}{4}\alpha^2+(m+k)\alpha}\left(\frac{\lambda_1\mu_2+\lambda_2\mu_1}{\mu_1+\mu_2} - (p_1+p_2)\int_0^\alpha e^{-(\frac{n}{4}s^2+(m+k)s)}ds\right). \tag{A5}$$

By definition we have $h(\alpha) \to 0$ as $\alpha \to \infty$. Using the above result we should have

$$p_1+p_2 = \frac{\lambda_1\mu_2+\lambda_2\mu_1}{\mu_1+\mu_2}\left(\int_0^\infty e^{-(\frac{n}{4}s^2+(m+k)s)}ds\right)^{-1}.$$

Now solving $\hat{g}_2(\alpha)$ in the definition of $h(\alpha)$ one gets $\hat{g}_2(\alpha) = \frac{h(\alpha)-\lambda_1\hat{g}_1(\alpha)}{\lambda_2}$. Substitute this result into the first differential equation in (A2) to have

$$\left(-\mu_1 - \frac{\lambda_1\mu_2}{\lambda_2} + \frac{\sigma_1^2}{2}\alpha^2 + (b_1+k\lambda_1)\alpha\right)\hat{g}_1(\alpha) - \lambda_1\alpha\hat{g}_1'(\alpha) = p_1\alpha - \frac{\mu_2}{\lambda_2}h(\alpha).$$

Divide both sides by $\lambda_1\alpha$, to get

$$\left(-(\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2})\frac{1}{\alpha} + \frac{\sigma_1^2}{2\lambda_1}\alpha + \frac{b_1}{\lambda_1} + k\right)\hat{g}_1(\alpha) - \hat{g}_1'(\alpha) = \frac{p_1}{\lambda_1} - \frac{\mu_2}{\lambda_1\lambda_2\alpha}h(\alpha).$$

Now to solve $\hat{g}_1(\alpha)$ one can apply the same method as we solve $h(\alpha)$. Hence the result is

$$\hat{g}_1(\alpha) = \alpha^{-(\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2})}e^{\frac{\sigma_1^2}{4\lambda_1}\alpha^2+(\frac{b_1}{\lambda_1}+k)\alpha}\int_0^\alpha\left(\frac{\mu_2}{\lambda_1\lambda_2 s}h(s) - \frac{p_1}{\lambda_1}\right)s^{\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}}e^{-\frac{\sigma_1^2}{4\lambda_1}s^2-(\frac{b_1}{\lambda_1}+k)s}ds. \tag{A6}$$

By symmetry of parameters, we have

$$\hat{g}_2(\alpha) = \alpha^{-(\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2})} e^{\frac{\sigma_2^2}{4\lambda_2}\alpha^2+(\frac{b_2}{\lambda_2}+k)\alpha} \int_0^\alpha \left(\frac{\mu_1}{\lambda_1\lambda_2 s} h(s) - \frac{p_2}{\lambda_2} \right) s^{\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}} e^{-\frac{\sigma_2^2}{4\lambda_2}s^2-(\frac{b_2}{\lambda_2}+k)s} ds. \tag{A7}$$

Notice that by definition $\hat{g}_i(\alpha) \to 0$ as $\alpha \to \infty$, thus the integral in $\hat{g}_i(\alpha)$ must have limit 0 as $\alpha \to \infty$, and by this property we can get p_1 and p_2:

$$p_1 = \frac{\mu_2}{\lambda_2} \int_0^\infty h(s) s^{\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}-1} e^{-\frac{\sigma_1^2}{4\lambda_1}s^2-(\frac{b_1}{\lambda_1}+k)s} ds \left(\int_0^\infty s^{\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}} e^{-\frac{\sigma_1^2}{4\lambda_1}s^2-(\frac{b_1}{\lambda_1}+k)s} ds \right)^{-1}, \tag{A8}$$

and

$$p_2 = \frac{\mu_1}{\lambda_1} \int_0^\infty h(s) s^{\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}-1} e^{-\frac{\sigma_2^2}{4\lambda_2}s^2-(\frac{b_2}{\lambda_2}+k)s} ds \left(\int_0^\infty s^{\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}} e^{-\frac{\sigma_2^2}{4\lambda_2}s^2-(\frac{b_2}{\lambda_2}+k)s} ds \right)^{-1}, \tag{A9}$$

Now, we are prepared to compute the moments of the stationary distribution of Y_t. To ease our calculation load, let's first do a change of variable by letting $s = u\alpha$ in the integral of $\hat{g}_i(\alpha)$. For example, $\hat{g}_1(\alpha)$ becomes

$$\hat{g}_1(\alpha) = e^{\frac{\sigma_1^2}{4\lambda_1}\alpha^2+(\frac{b_1}{\lambda_1}+k)\alpha} \int_0^1 \left(\frac{\mu_2}{\lambda_1\lambda_2} h(u\alpha) - \frac{p_1}{\lambda_1} u\alpha \right) u^{\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}-1} e^{-\frac{\sigma_1^2}{4\lambda_1}(u\alpha)^2-(\frac{b_1}{\lambda_1}+k)u\alpha} du. \tag{A10}$$

Then it's straightforward to find that

$$
\begin{aligned}
\hat{g}_1'(\alpha) = {} & (\frac{\sigma_1^2}{2\lambda_1}\alpha + \frac{b_1}{\lambda_1} + k) e^{\frac{\sigma_1^2}{4\lambda_1}\alpha^2 + (\frac{b_1}{\lambda_1}+k)\alpha} \\
& \times \int_0^1 \left(\frac{\mu_2}{\lambda_1\lambda_2} h(u\alpha) - \frac{p_1}{\lambda_1} u\alpha \right) u^{\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} - 1} e^{-\frac{\sigma_1^2}{4\lambda_1}(u\alpha)^2 - (\frac{b_1}{\lambda_1}+k)u\alpha} du \\
& + e^{\frac{\sigma_1^2}{4\lambda_1}\alpha^2 + (\frac{b_1}{\lambda_1}+k)\alpha} \int_0^1 \left(\frac{\mu_2}{\lambda_1\lambda_2} u h'(u\alpha) - \frac{p_1}{\lambda_1} u \right) u^{\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} - 1} e^{-\frac{\sigma_1^2}{4\lambda_1}(u\alpha)^2 - (\frac{b_1}{\lambda_1}+k)u\alpha} du \\
& - e^{\frac{\sigma_1^2}{4\lambda_1}\alpha^2 + (\frac{b_1}{\lambda_1}+k)\alpha} \\
& \times \int_0^1 \left(\frac{\mu_2}{\lambda_1\lambda_2} h(u\alpha) - \frac{p_1}{\lambda_1} u\alpha \right) u^{\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} - 1} \left(\frac{\sigma_1^2}{2\lambda_1} u^2\alpha + (\frac{b_1}{\lambda_1} + k)u \right) e^{-\frac{\sigma_1^2}{4\lambda_1}(u\alpha)^2 - (\frac{b_1}{\lambda_1}+k)u\alpha} du.
\end{aligned}
\tag{A11}
$$

To simplify, let $\theta = \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2}$, and $p(k) = p_1(k) + p_2(k)$, and recall that $b_i = m\lambda_i$, one gets

$$
\begin{aligned}
\hat{g}_1'(0) &= \frac{1}{\theta+1} \left(\frac{\mu_2}{\mu_1+\mu_2} \left(\frac{b_1}{\lambda_1} + m\theta + (1+\theta)k \right) - \frac{\mu_2}{\lambda_1\lambda_2} p(k) - \frac{p_1(k)}{\lambda_1} \right) \\
&= \frac{1}{\theta+1} \left(\frac{\mu_2}{\mu_1+\mu_2} (m+k)(1+\theta) - \frac{\mu_2}{\lambda_1\lambda_2} p(k) - \frac{p_1(k)}{\lambda_1} \right).
\end{aligned}
\tag{A12}
$$

By symmetry of parameters one also has

$$
\begin{aligned}
\hat{g}_2'(0) &= \frac{1}{\theta+1} \left(\frac{\mu_1}{\mu_1+\mu_2} \left(\frac{b_2}{\lambda_2} + m\theta + (1+\theta)k \right) - \frac{\mu_1}{\lambda_1\lambda_2} p(k) - \frac{p_2(k)}{\lambda_2} \right) \\
&= \frac{1}{\theta+1} \left(\frac{\mu_1}{\mu_1+\mu_2} (m+k)(1+\theta) - \frac{\mu_1}{\lambda_1\lambda_2} p(k) - \frac{p_2(k)}{\lambda_2} \right).
\end{aligned}
\tag{A13}
$$

Hence

$$
\begin{aligned}
\mathbb{E}\,(X_\infty) &= \mathbb{E}\,(Y_\infty) + k \\
&= -\hat{g}_1'(0) - \hat{g}_2'(0) + k \\
&= \frac{1}{\theta+1} \left(\frac{\mu_1+\mu_2}{\lambda_1\lambda_2} p(k) + \frac{p_1(k)}{\lambda_1} + \frac{p_2(k)}{\lambda_2} - m(1+\theta) \right).
\end{aligned}
\tag{A14}
$$

Likewise, other moments can be achieved if we further the calculations. From (A10) we can get

$$\begin{aligned}\hat{g}_1''(0) = &\frac{\sigma_1^2}{\lambda_1}\frac{\mu_2}{\mu_1+\mu_2}\frac{1}{\theta+2} + \frac{2}{(\theta+1)(\theta+2)}\frac{\mu_2}{\mu_1+\mu_2}\left(\frac{b_1}{\lambda_1}+k\right)^2 \\ &+ \frac{2}{(\theta+1)(\theta+2)}\left(\frac{b_1}{\lambda_1}+k\right)\left(\frac{\mu_2\theta}{\mu_1+\mu_2}(m+k) - \frac{\mu_2}{\lambda_1\lambda_2}p_1(k) - \frac{p_1(k)}{\lambda_1}\right) \\ &+ \frac{1}{\theta+2}\left(\frac{\mu_2\theta}{\mu_1+\mu_2}(\frac{2}{n}+(m+k)^2) - \frac{\mu_2}{\lambda_1\lambda_2}p_1(k)(m+k)\right).\end{aligned} \tag{A15}$$

Noticing the symmetry of parameters, $\hat{g}_2''(0)$ will be a straightforward exchange of parameters. Recall that $b_i = m\lambda_i$ and $\sigma_i = \sqrt{n\lambda_i}$. Now we have

$$\begin{aligned}\mathbb{E}\,(Y_\infty^2) &= \hat{g}_1''(0)+\hat{g}_2''(0) \\ &= \frac{n}{2}+(m+k)^2 - \frac{2}{(\theta+1)(\theta+2)}(m+k)\left(\frac{p_1(k)}{\lambda_1}+\frac{p_2(k)}{\lambda_2}\right) \\ &- \frac{\theta+3}{(\theta+1)(\theta+2)}(m+k)\frac{\mu_1+\mu_2}{\lambda_1\lambda_2}p(k).\end{aligned} \tag{A16}$$

Combining the above result with (A14) we can obtain that

$$\begin{aligned}Var(X_\infty) = Var(Y_\infty) &= \mathbb{E}\,(Y_\infty^2) - (\mathbb{E}\,Y_\infty)^2 \\ &= (\hat{g}_1''(0)+\hat{g}_2''(0)) - (-\hat{g}_1'(0)-\hat{g}_2'(0)) \\ &= \frac{n}{2}+(m+k)^2 - \frac{2}{(\theta+1)(\theta+2)}(m+k)\left(\frac{p_1(k)}{\lambda_1}+\frac{p_2(k)}{\lambda_2}\right) \\ &- \frac{\theta+3}{(\theta+1)(\theta+2)}(m+k)\frac{\mu_1+\mu_2}{\lambda_1\lambda_2}p(k) \\ &- \frac{1}{(\theta+1)^2}\left(\frac{\mu_1+\mu_2}{\lambda_1\lambda_2}p(k)+\frac{p_1(k)}{\lambda_1}+\frac{p_2(k)}{\lambda_2}-m(1+\theta)\right)^2.\end{aligned} \tag{A17}$$

In the same manner, we can get $\hat{g}_1^{(3)}(0)$ and $\hat{g}_2^{(3)}(0)$, and

$$\begin{aligned}\hat{g}_1^{(3)}(0)+\hat{g}_2^{(3)}(0) = &(m+k)^3 - \frac{\theta^2+6\theta-1}{(\theta+1)(\theta+2)(\theta+3)}\frac{\mu_1+\mu_2}{\lambda_1\lambda_2}(m+k)^2 p(k) \\ &+\theta\left(\frac{n}{2(\theta+4)}+\frac{1}{\theta+2}\right)(m+k)^2+\frac{n(\theta+9)}{2(\theta+3)}(m+k) \\ &-\left(\frac{n}{2(\theta+4)}+\frac{1}{\theta+2}\right)\frac{\mu_1+\mu_2}{\lambda_1\lambda_2}(m+k)p(k)-\frac{n}{2(\theta+4)}(m+k)\left(\frac{p_1(k)}{\lambda_1}+\frac{p_2(k)}{\lambda_2}\right) \\ &-\frac{n(\theta+4)}{(\theta+1)(\theta+3)}\frac{\mu_1+\mu_2}{\lambda_1\lambda_2}p(k)-\frac{6}{(\theta+1)(\theta+2)}\left(\frac{p_1(k)}{\lambda_1}+\frac{p_2(k)}{\lambda_2}\right)\end{aligned} \tag{A18}$$

and thus we can obtain $\mathbb{E}\,(X_\infty^3)$ by the following computation:

$$\begin{aligned}\mathbb{E}\,(X_\infty^3) &= \mathbb{E}\,(Y_\infty+k)^3 \\ &= \mathbb{E}\,(Y_\infty^3)+3k\mathbb{E}\,(Y_\infty^2)+3k^2\mathbb{E}\,(Y_\infty)+k^3 \\ &= -\hat{g}_1^{(3)}(0)-\hat{g}_2^{(3)}(0)+3k(\hat{g}_1''(0)+\hat{g}_2''(0))+3k^2(-\hat{g}_1'(0)-\hat{g}_2'(0))+k^3.\end{aligned} \tag{A19}$$

6.3 Utility Optimizations

A. Ratio Utility

We first consider the type of utility functions in the following form:

$$U_1(k) = \frac{\mathbb{E}\,h(X_\infty)}{f(Var(X_\infty))} \tag{A1}$$

where h is a given integrable function on $[k,\infty]$ such that $\mathbb{E}\,h(X_\infty)$ exists. f is introduced to turn the ratio $U_1(k)$ to be unit free.

As the motivations mentioned in the Introduction, one applicable situation of $U_1(k)$ would be in stock markets. X_t could stand for a listed company's stock price, and h

is some function translating X_t into the book value of the company, with J_t being the regime-switching between 'Bull' and 'Bear' markets. In this case $U_1(k)$ naturally defines a measure for the value of this company as putting its book value in numerator and volatility (risk) in denominator, and thus its maximum represents the highest value this company can achieve per unit risk. Thus for a stock market where a regulating authority defines lower daily price limit such as Shanghai Stock Exchange or Tokyo Stock Exchange, our model allows the authority to determine the optimal lower daily price limit for a listed company which maximizes its value in terms of $U_1(k)$.

Case $h(x) = x$

Let's start with a simple case: $h(x) = x$. Thus we should have $f(x) = \sqrt{x}$ and $U_1(k) = \frac{\mathbb{E}\ (X_\infty)}{\sqrt{Var(X_\infty)}}$, i.e., the ratio of the stationary expected stock price to its standard deviation.

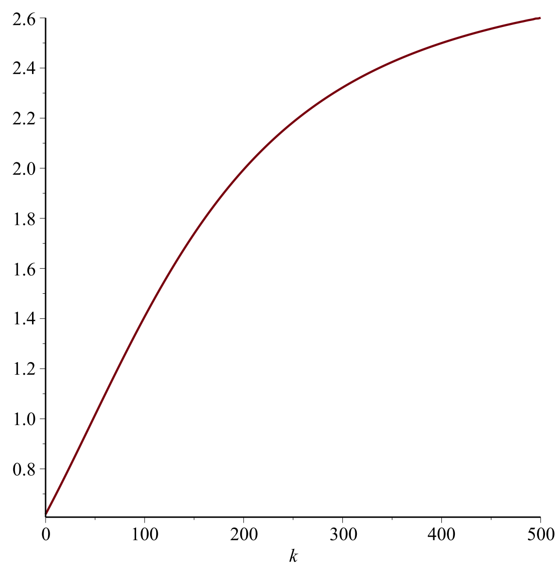

Figure 6.1: Case $h(x) = x$. Plot of $U_1(k)$ with
$\mu_1 = 0.1, \mu_2 = 0.2, \lambda_1 = 0.1, \lambda_2 = 0.2, m = 1, n = 10^4$.

In this case one can see that the utility ratio U_1 increases as k increases. We can conclude that there is no optimal lower price limit k, and obviously the higher value of k, the better. To further our discussions, first note that the case $h(x) = x^2$ make little sense owing to the fact that the second moment is different from the variance only by a constant. Therefore we directly move to the cubic case of h.

Case $h(x) = x^3$

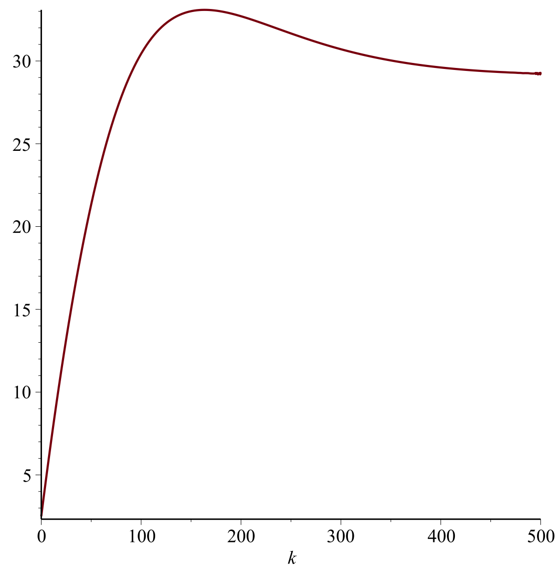

Figure 6.2: Case $h(x) = x^3$. Plot of $U_1(k)$ with
$\mu_1 = 0.1, \mu_2 = 0.2, \lambda_1 = 0.1, \lambda_2 = 0.2, m = 1, n = 10^4$.

As $h(x) = x^3$, we should have $f(x) = x^{\frac{3}{2}}$ and $U_1(k) = \frac{\mathbb{E}\,(X_\infty^3)}{Var(X_\infty)^{\frac{3}{2}}}$. In this case U_1 does achieve the optimal lower price limit at $k^* = 163.5280262$.

B. Difference Utility

Secondly let's consider the utility function:

$$U_2(k) = \alpha \mathbb{E}\, h(X_\infty) - \beta f(Var(X_\infty)) \tag{A2}$$

where h and f are the same as in the definitions of U_1. $\alpha, \beta > 0$ are given constants such that $\alpha + \beta = 1$, and they express the preference of the policy maker (regulating authority) between value and risk. A risk-averse authority could choose a high β and a low α to measure a listed company's quality, while a risk-seeking authority might choose the opposite combination of α and β.

Case $h(x) = x$

Again we shall start from a simple case, $h(x) = x$, and in this case $f(x) = \sqrt{x}$. Namely U_2 is the weighted difference between the stationary expected stock price and its standard deviation. Furthermore let's assume a risk-averse authority is using $U_2(k)$ to make decisions. Thus in this case let's set $\alpha = 0.35, \beta = 0.65$.

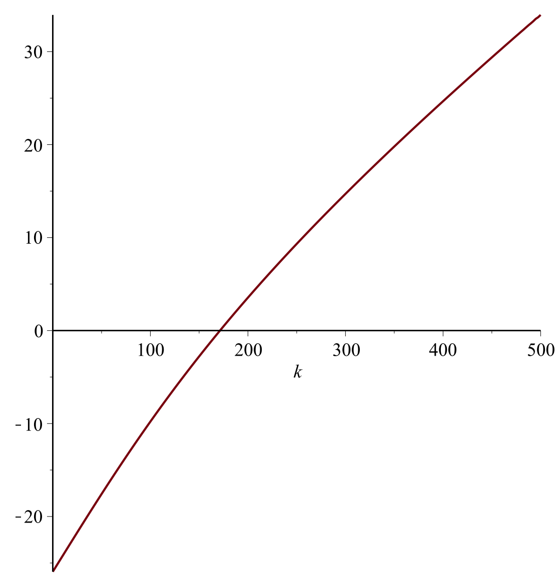

Figure 6.3: Case $h(x) = x$. Plot of $U_2(k)$ with
$\mu_1 = 0.1, \mu_2 = 0.2, \lambda_1 = 0.1, \lambda_2 = 0.2, m = 1, n = 10^4$.

First we can notice that the difference utility is an increasing function in k, and so the higher k, the better. Also a reasonable authority should force the lower price limit k

to be bigger than 171.3770147 to avoid a negative utility.

Like before the case $h(x) = x^2$ does not make much sense so we will jump to the cubic h case.

Case $h(x) = x^3$

Secondly, we try the case $h(x) = x^3$, $f(x) = x^{\frac{3}{2}}$ with an even more risk-averse authority having $\alpha = 0.033, \beta = 0.967$:

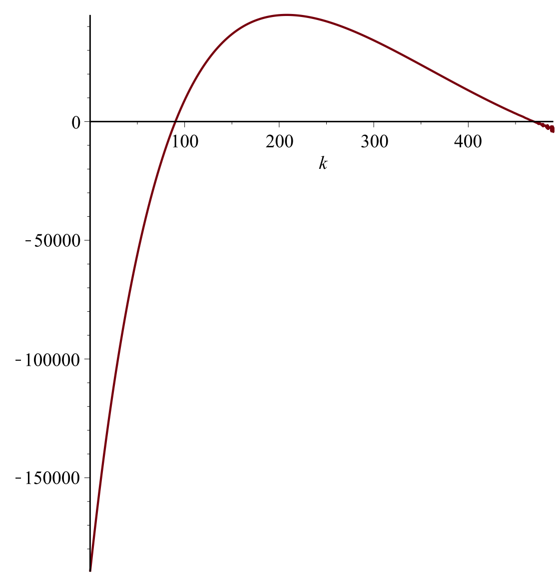

Figure 6.4: Case $h(x) = x^3$. Plot of $U_2(k)$ with
$\mu_1 = 0.1, \mu_2 = 0.2, \lambda_1 = 0.1, \lambda_2 = 0.2, m = 1, n = 10^4$.

First, to obtain a positive utility, the authority should determine the lower price limit k within the interval $[90.25518131, 469.9707131]$. And in this case the optimal lower price limit can be attained at $k^* = 208.1253977$.

Chapter 7

Asymptotically optimal admission control of queueing models with impatient costumers and performance penalty functions

7.1 Motivations and model formulation

Stochastic optimal control problems for various types of queueing models have been studied intensively in decades. It is natural to consider those systems with impatient customers, since the well observed phenomenon is a customer will renege, i.e., abandon the queue if his or her service has not begun within a certain amount of time. For one instance, we refer our reader to Ward and Kumar's paper [25] and the reference list therein. Inspired by the methodology in their paper, we propose a queueing model with above features plus a penalty function which incorporates the performance evaluation of servers. To be specific, the Yerkes-Dodson law says that the interplay between performance (or efficiency) and workload of a human server often follows a bell-shaped curve. Namely, performance increases with workload (when workload is not enough and servers experience idletime), but only up to a point, and after passing that performance decreases as workload gets higher (when tasks at hand become too stressful rather than incentive for servers). Based on this, we assume the existence of a unique performance maximizer q, and then a penalty function g that penalties more as the queue length becomes further form q.

With the above setting up, our goal is to find an optimal and stationary admission policy for the system manager to decide whether to admit or deny an arriving customer, in order to minimize the expected infinite-horizon discounted cost. Mathematically, we are looking for an optimal barrier b^*, and the corresponding admission control policy would be a function π defined on the set of nonnegative integers such that $\pi(x) := 1\{x < b^*\}$. Now let's introduce the definitions and notations below:

Given a filtered probability space $\Lambda := (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with the filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfying the *usual* conditions, the triple $((Z, L, R, W), (\Omega, \mathscr{F}, \mathbb{P}), \{(\mathscr{F}_t)_{t\geq 0}\})$ is a weak solution to:

$$\begin{cases} Z_t = W_t - \gamma \int_0^t Z_s ds + L_t - R_t, \\ Z_t \in [0, b], \end{cases} \tag{A1}$$

where Z_t is the approximating diffusion representing queue length, W_t the standard Brownian motion with $W_0 = x$, and γ is the customer reneging rate. As in previous chapters, L_t and R_t are the adapted, nonnegative, nondecreasing reflected processes confining Z_t such that $Z_t \geq 0$ and $Z_t \leq b$, respectively.

The discounted cost on infinite-horizon is defined as:

$$u(x) = \mathbb{E}_x \left[\int_0^\infty e^{-\delta t} (pdR_t + r\gamma Z_t dt + g(Z_t)dt) \right] \tag{A2}$$

where g is a convex function satisfying integrable conditions, and δ is the compound interest rate. The cost function u penalties the system manager p dollars when a coming customer is denied, r dollars when a customer reneges, and $g(Z_t)$ when queue length is far from performance maximizer q.

Proceeding as Proposition 3.1 and 3.2 in [25], by Itô's formula, one can get the following two propositions.

Proposition 7.1. *A function $u(x)$ satisfying*

$$\begin{cases} \dfrac{\sigma^2}{2}u''(x) - \gamma x u'(x) - \delta u(x) + r\gamma x + g(x) = 0, \quad 0 \leq x \leq b, \\ u'(0) = 0, \quad u'(b) = p \end{cases} \tag{A3}$$

is the expected infinite-horizon discounted cost in (A2).

Proposition 7.2. *if there exists a function $u_*(x)$ with associated barrier at b^* satisfying*

$$\begin{cases} \dfrac{\sigma^2}{2}u''(x) - \gamma x u'(x) - \delta u(x) + r\gamma x + g(x) \geq 0, \quad x \geq 0, \\ u'(0) = 0, \quad \text{and} \quad u'(x) \leq p \quad \text{for all } x \geq 0\,. \end{cases} \tag{A4}$$

Then for any $x \geq 0$, we have

$$u_*(x) \leq \mathbb{E}_x \left[\int_0^\infty e^{-\delta t} (p dR_t + r\gamma R_t dt + g(Z_t) dt) \right].$$

for any admissible control policy R in (A1).

The above two conclusions provide us guidelines for searching for the optimal admission barrier b^*, since they characterizes u as a solution to an ODE and also point out a criterion to pick the optimal admission policy.

7.2 An Iterative strategy to locate b^*

A. Case 1: $g(x) = (x-q)^2$

In order to show an analytically tractable solution, let's assume $g(x) = (x-q)^2$. Similar to [25], we can develop a iterative scheme to locate the optimal barrier b^*.

Let J be the Kummer's series defined by:

$$J(a,b;x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!}$$

where $(a)_k = a(a+1)...(a+k-1)$ with $(a)_0 = 1$. Now the following b-dependent function $h(x)$ is the solution to (A3):

$$h(x) = J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}x^2)\left(C_1(b) + C_2 \int_0^x \frac{e^{\frac{\gamma}{\sigma^2}y^2}}{J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}y^2)^2} dy\right) + s(x) \qquad \text{(A1)}$$

with

$$C_2 = -\frac{r\gamma - 2q}{\gamma + \delta},$$

$$C_1(b) = \frac{r\gamma - 2q}{\gamma + \delta}\left(\int_0^b \frac{e^{\frac{\gamma}{\sigma^2}y^2}}{J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}y^2)^2} dy + \frac{\sigma^2}{2\delta b} \frac{1}{J(\frac{\delta}{2\gamma}+1, \frac{3}{2}; \frac{\gamma}{\sigma^2}b^2)} \left(\frac{e^{\frac{\gamma}{\sigma^2}b^2}}{J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}b^2)} - 1\right)\right)$$
$$+ \sigma^2 \left(\frac{p}{2\delta b} - \frac{1}{\delta(2\gamma+\delta)}\right) \frac{1}{J(\frac{\delta}{2\gamma}+1, \frac{3}{2}; \frac{\gamma}{\sigma^2}b^2)},$$

$$s(x) = \frac{1}{2\gamma+\delta}x^2 + \frac{r\gamma - 2q}{\gamma+\delta}x + \frac{1}{\delta}(q^2 + \frac{\sigma^2}{2\gamma+\delta}).$$

We obtain this by first getting the solution to the corresponding homogeneous equation in (A3) using Equation (108) in Polyanin and Zaitsev [22]. Then use Formula (2) in 2.1.1 in [22] we obtain the analytic solution to (A3). Finally applying the property from

Slater in [24]:

$$\frac{d}{dx}J(a,b,x) = \frac{a}{b}J(a+1,b+1,x),$$

and by the method of undetermined coefficients we can get $C_1(b)$ and C_2.

Now define

$$u_b(x) = \begin{cases} h(x), & 0 \le x \le b \\ p(x-b) + h(b), & b < x \end{cases} \tag{A2}$$

If we can find a $b^* > 0$ associated with $u_*(x)$ that satisfies $u_*''(b^*) = 0$, then by Prop 7.2, b^* is the optimal barrier. Starting from some b_0 such that $u_{b_0}''(b_0) < 0$, the iterative scheme below can locate the optimal barrier b^*:

$$b_{n+1} = \max\{b \in [0,b_n], b \textit{ maximizes } u_n' \textit{ on } [0,b_n]\}. \tag{A3}$$

To see this, First we claim that there exists b_0 such that $u_0''(b_0) < 0$. Straight Substitution of b into (A2) shows that

$$\begin{aligned} u_b(b) = &\frac{J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}b^2)}{J(\frac{\delta}{2\gamma}+1, \frac{3}{2}; \frac{\gamma}{\sigma^2}b^2)} \left(\frac{\sigma^2(r\gamma - 2q)}{2\delta b(\gamma+\delta)} \left(e^{\frac{\gamma}{\sigma^2}b^2} J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}b^2) - 1 \right) + \sigma^2 \left(\frac{p}{2\delta b} - \frac{1}{\delta(2\gamma+\delta)} \right) \right) \\ &+ \frac{1}{2\gamma+\delta}b^2 + \frac{r\gamma - 2q}{\gamma+\delta}b + \frac{1}{\delta}(q^2 + \frac{\sigma^2}{2\gamma+\delta}). \end{aligned} \tag{A4}$$

Now use Formula (13.1.4) in Slater [24],

$$J(a,b,x) = \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b} \left(1 + O(|x|^{-1})\right), \quad \text{as} \quad |x| \to \infty$$

we can get

$$\frac{u_b(b)}{b^2} \to \frac{1}{2\gamma+\delta}, \quad \text{as} \quad b \to \infty. \tag{A5}$$

Thus by (A3) we have

$$\frac{\sigma^2}{2}u_b''(b) = \gamma b u_b'(b) + \delta u_b(b) - r\gamma b - (b-q)^2$$
$$= (p-r)\gamma b + \delta u_b(b) - (b-q)^2,$$

and combine this with (A5) one can see

$$\frac{u_b''(b)}{b^2} \to \frac{2}{\sigma^2}\left(\frac{\delta}{2\gamma+\delta} - 1\right) < 0, \quad \text{as} \quad b \to \infty,$$

hence

$$u_b''(b) \to -\infty, \quad \text{as} \quad b \to \infty,$$

and there exists a b_0 such that $u_0''(b_0) < 0$. The rest of needed proof is similar to Proposition 3.3 in [25], and then we can conclude that $b_n \to b^*$, as $n \to \infty$.

Lemma 7.3. *Under the assumption $p/r < (1+\delta/\gamma)^{-1}$ and $0 < q < b^*$, the barrier at b^* is optimal, and*

$$(p-r)\gamma b^* + \delta u_*(b^*) - (b^*-q)^2 = 0 \tag{A6}$$

To prove b^* is the optimal barrier, based on Prop 7.2 it only remains to show that for $x > b^*$,

$$\frac{\sigma^2}{2}u_*''(x) - \gamma x u_*'(x) - \delta u_*(x) + r\gamma x + g(x) \geq 0.$$

Recall that for $x > b^*$, $u_*(x) = p(x-b^*) + h(b^*)$, so we have $u_*''(x) = 0$ and $u_*'(x) = p$, and thus by (A6) and assumptions in the above lemma,

$$
\begin{aligned}
&\frac{\sigma^2}{2}u_*''(x) - \gamma x u_*'(x) - \delta u_*(x) + r\gamma x + g(x) \\
&= -\gamma p x - \delta(p(x-b^*) + h(b^*)) + r\gamma x + (x-q)^2 \\
&= -\gamma p x - \delta p(x-b^*) - (r-p)\gamma b^* - (b^*-q)^2 + r\gamma x + (x-q)^2 \\
&= (x-b^*)(\gamma(r-p) - \delta p) + (x-q)^2 - (b^*-q)^2 \\
&\geq 0
\end{aligned}
$$

B. Case 2: $g(x) = |x-q|$

Secondly, consider $g(x) = |x-q|$ with assumption $p < \frac{r\gamma+1}{\gamma+\delta}$. Then we can define

$$
u_b(x) = \begin{cases}
u_1(x) = J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}x^2)\left(C_1(b) + C_2\int_0^x \frac{e^{\frac{\gamma}{\sigma^2}y^2}}{J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}y^2)^2}dy\right) + \frac{r\gamma-1}{\gamma+\delta}x + \frac{q}{\delta}, & 0 < x \leq q \\
u_2(x) = J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}x^2)\left(C_3(b) + C_4\int_0^x \frac{e^{\frac{\gamma}{\sigma^2}y^2}}{J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}y^2)^2}dy\right) + \frac{r\gamma+1}{\gamma+\delta}x - \frac{q}{\delta}, & q < x \leq b \\
p(x-b) + u_b(b), & b < x
\end{cases}
\tag{A7}
$$

By the boundary conditions in (A3), we know $u_1'(0) = 0$ and $u_2'(b) = p$. With assumptions $u_1(q) = u_2(q)$ and $u_1'(q) = u_2'(q)$, we can find the four constants above. First to simplify the notations, let us define

$$
\begin{aligned}
J_1(x) &= J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}x^2), \\
J_2(x) &= J(\frac{\delta}{2\gamma} + 1, \frac{3}{2}; \frac{\gamma}{\sigma^2}x^2), \\
I(x) &= \int_0^x \frac{e^{\frac{\gamma}{\sigma^2}y^2}}{J(\frac{\delta}{2\gamma}, \frac{1}{2}; \frac{\gamma}{\sigma^2}y^2)^2}dy.
\end{aligned}
$$

Then we can get:

$$
\begin{aligned}
C_2 &= -\frac{r\gamma-1}{\gamma+\delta},\\
C_4 &= -C_2 - e^{-\frac{\gamma}{\sigma^2}q^2}\left(\frac{4\gamma q^2 J_2(q)}{\sigma^2(\gamma+\delta)} + \frac{2}{(\gamma+\delta)J_1(q)}\right),\\
C_3(b) &= \frac{\sigma^2}{2\delta b J_2(b)}\left(p - \frac{r\gamma+1}{\gamma+\delta} - C_4\frac{e^{\frac{\gamma}{\sigma^2}b^2}}{J_1(b)}\right) - C_4 I(b),\\
C_1(b) &= C_3(b) - (C_2 - C_4)I(q) - \frac{2\gamma q}{\delta(\gamma+\delta)J_1(q)}.
\end{aligned}
$$

Then substitution for constants $C_3(b)$ and C_4 yields:

$$
u_b(b) = \frac{\sigma^2}{2\delta b}\frac{J_1(b)}{J_2(b)}\left(p - \frac{r\gamma+1}{\gamma+\delta} - C_4\frac{e^{\frac{\gamma}{\sigma^2}b^2}}{J_1(b)}\right) + \frac{r\gamma+1}{\gamma+\delta}b - \frac{q}{\delta}. \tag{A8}
$$

And it can be shown that the iterative strategy A3 still works here.

Again by the property

$$
J(a,b,x) = \frac{\Gamma(b)}{\Gamma(a)}e^x x^{a-b}\left(1 + O(|x|^{-1})\right), \quad \text{as} \quad |x| \to \infty
$$

we obtain

$$
\frac{u_b(b)}{b} \to \frac{r\gamma+1}{\gamma+\delta}, \quad \text{as} \quad b \to \infty. \tag{A9}
$$

Now by the differential equation in (A3) we have

$$
\begin{aligned}
\frac{\sigma^2}{2}u_b''(b) &= \gamma b u_b'(b) + \delta u_b(b) - r\gamma b - |b-q|\\
&= (p-r)\gamma b + \delta u_b(b) - (b-q),
\end{aligned}
$$

combining this with the condition $p < \frac{r\gamma+1}{\gamma+\delta}$, it implies that

$$\frac{u_b''(b)}{b} \to \frac{2\gamma}{\sigma^2}\left(p - \frac{r\gamma+1}{\gamma+\delta}\right) < 0, \quad \text{as} \;\; b \to \infty,$$

hence

$$u_b''(b) \to -\infty, \quad \text{as} \;\; b \to \infty,$$

and there exists a b_0 such that $u_0''(b_0) < 0$. Again we can finish the needed proof as Proposition 3.3 in [25], and then we can conclude that $b_n \to b^*$, as $b \to \infty$.

Lemma 7.4. *Under the assumption* $p < \frac{r\gamma+1}{\gamma+\delta}$ *and* $0 < q < b^*$*, the barrier at* b^* *is optimal, and*

$$b^* = \frac{\delta u_*(b^*) + q}{\gamma(r-p)+1} \tag{A10}$$

Again like before it suffices to show that for $x > b^*$,

$$\frac{\sigma^2}{2}u_*''(x) - \gamma x u_*'(x) - \delta u_*(x) + r\gamma x + g(x) \geq 0.$$

Recall that for $x > b^*$, $u_*(x) = p(x-b^*) + u_*(b^*)$, so we have $u_*''(x) = 0$ and $u_*'(x) = p$, and thus by (A10) and condition $p < \frac{r\gamma+1}{\gamma+\delta}$ in the above lemma,

$$\begin{aligned}
&\frac{\sigma^2}{2}u_*''(x) - \gamma x u_*'(x) - \delta u_*(x) + r\gamma x + g(x) \\
&= -\gamma p x - \delta(p(x-b^*) + u_*(b^*)) + r\gamma x + |x-q| \\
&= -\gamma p x - \delta p(x-b^*) - b^*((r-p)\gamma+1) + q + r\gamma x + x - q \\
&= x(r\gamma+1-p(\gamma+\delta)) - b^*(r\gamma+1-p(\gamma+\delta)) \\
&= (x-b^*)(r\gamma+1-p(\gamma+\delta)) \\
&\geq 0
\end{aligned}$$

Chapter 8

Optimal control policy for a linear-quadratic regulator problem

8.1 Introduction

We consider a class of one-dimensional reflected stochastic differential equations (SDEs) with one barrier. A typical quadratic cost problem on a finite time horizon $[0,T]$ is studied. Theoretical procedures and results are present here.

8.2 Model formulation and solving

Let us first introduce our reflected SDE model . Given a filtered probability space $\Lambda := (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with the filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfying the *usual* conditions, we are concerned with the strong solution $\{X_t : t \geq 0\}$ of the following reflected SDE with one-sided barrier 0:

$$\begin{cases} dX_t = (a_t X_t + b_t u_t)dt + (c_t X_t + d_t u_t)dB_t + dL_t, \\ X_0 = x \in [0, \infty). \end{cases} \tag{A1}$$

Here, a_t, b_t, c_t, d_t are given functions in t satisfying necessary integrable conditions. u_t is our control variable, the policy applied to the process. The process $L = (L_t)_{t\geq 0}$ is the well-known minimal non-decreasing and non-negative process, which keeps the process $X_t \geq 0$ for all $t \geq 0$ with the minimal effort. More precisely, the process L increases only when X hits the boundary 0, respectively, so that

$$\int_0^\infty 1_{\{X_t>0\}} dL_t = 0 \quad \text{and} \quad L_0 = 0,$$

where $1(\cdot)$ is the indicator function.

Given the above reflected diffusions, now we introduce our quadratic cost structure C as below:

$$C(u_t) = \mathbb{E}\left[\int_0^T (Q_t X_t^2 + R_t u_t^2) dt + H_T X_T^2\right], \tag{A2}$$

where Q_t, R_t and H_t are all t-dependent functions satisfying necessary integrable conditions. A natural task is to find the best suitable policy u_t which minimizes the cost.

Now suppose P_t is a differentiable function in t, satisfying the following ODE:

$$\begin{cases} \dot{P}_t = \tilde{a}_t P_t^2 + \tilde{b}_t P_t - Q_t, \\ P_T = H_T, \end{cases} \tag{A3}$$

where

$$\tilde{a}_t = \frac{(b_t + c_t d_t)^2}{R_t + P_t d_t^2} \quad \text{and} \quad \tilde{b}_t = -2a_t - c_t^2.$$

Notice that this is a so-called Riccati equation, hence the solution exists and is unique. Using Itô's formula to $P_tX_t^2$ on $[0,T]$, we can get

$$\begin{aligned} P_TX_T^2 = &\int_0^T X_t^2\dot{P}_t dt + 2\int_0^T P_tX_t(a_tX_t + b_tu_t)dt \\ &+ 2\int_0^T P_tX_t(c_tX_t + d_tu_t)dB_t + 2\int_0^T P_tX_t dL_t + \int_0^T P_t(c_tX_t + d_tu_t)^2 dt. \end{aligned} \tag{A4}$$

Recall that $P_T = H_T$. By the above we can deduce the cost function $C(U_t)$ as

$$\begin{aligned} C(U_t) &= \mathbb{E}\left[\int_0^T (Q_tX_t^2 + R_tu_t^2) + X_t^2\dot{P}_t + 2X_tP_t(a_tX_t + b_tu_t) + P_t(c_tX_t + d_tu_t)^2 dt\right] \\ &= \mathbb{E}\left[\int_0^T (\tilde{Q}_tX_t^2 + \tilde{R}_tu_t^2 + G_tu_tX_t)dt\right] \\ &= \mathbb{E}\left[\int_0^T \tilde{R}_t\left(u_t + \frac{G_tX_t}{2\tilde{R}_t}\right)^2 - \frac{(G_tX_t)^2}{4\tilde{R}_t} + \tilde{Q}_tX_t^2 dt\right] \end{aligned} \tag{A5}$$

with

$$\begin{aligned} \tilde{Q}_t &= Q_t + \dot{P}_t + 2a_tP_t + P_tc_t^2, \\ \tilde{R}_t &= R_t + P_td_t^2, \\ \text{and}\quad G_t &= 2P_t(b_t + c_td_t). \end{aligned}$$

Now by (A3), only the squared term in $C(U_t)$ left and we finally have

$$C(U_t) = \mathbb{E}\left[\int_0^T \tilde{R}_t\left(u_t + \frac{G_tX_t}{2\tilde{R}_t}\right)^2 dt\right]$$

Thus clearly our best control policy which minimizes C should be

$$u_t^* = -\frac{G_tX_t}{2\tilde{R}_t}.$$

8.3 Appendix

Solving A3

Observe that

$$\begin{cases} \dot{P}_t = \tilde{a}_t P_t^2 + \tilde{b}_t P_t - Q_t , \\ P_T = H_T , \end{cases}$$

is a typical Riccati's equation. If $\tilde{a}_t$, $\tilde{b}_t$, and Q_t are all real numbers, then we could apply the method of separation of variables to solve. Otherwise, let's first assume we have a particular solution φ_t for this ODE (it can be obtained by many ways). Now we consider $\phi_t = P_t - \varphi_t$. Since both P_t and φ_t satisfy the above differential equation, a subtraction gives us:

$$\begin{aligned} \phi_t' = (P_t - \varphi_t)' &= \tilde{a}_t (P_t^2 - \varphi_t^2) + \tilde{b}_t (P_t - \varphi_t) \\ &= \tilde{a}_t (P_t - \varphi_t)(P_t + \varphi_t) + \tilde{b}_t (P_t - \varphi_t) \\ &= \tilde{a}_t \phi_t (\phi_t + 2\varphi_t) + \tilde{b}_t \phi_t \\ &= \tilde{a}_t \phi_t^2 + (2\tilde{a}_t \varphi_t + \tilde{b}_t)\phi_t . \end{aligned}$$

Now divide $-\phi_t^{-2}$ on both sides, we can get

$$\frac{d\phi_t^{-1}}{dt} + (2\tilde{a}_t \varphi_t + \tilde{b}_t)\phi_t^{-1} + \tilde{a}_t = 0.$$

And with boundary condition $\phi_T^{-1} = (H_T - \varphi_T)^{-1}$, the solution to this ODE is:

$$\phi_t^{-1} = \frac{1}{H_T - \varphi_T} e^{\int_t^T (2\tilde{a}_s \varphi_s + \tilde{b}_s) ds} + \int_t^T \tilde{a}_s e^{\int_t^s (2\tilde{a}_r \varphi_r + \tilde{b}_r) dr} ds .$$

So ϕ_t can be obtained by taking the reciprocal. And thus follows the solution to (A3) $P_t = \phi_t + \varphi_t$.

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